1 Functions

If A equals success, then the formula is A equals X plus Y plus Z, with X being work, Y play, and Z keeping your mouth shut.

Albert Einstein (1879 – 1955)

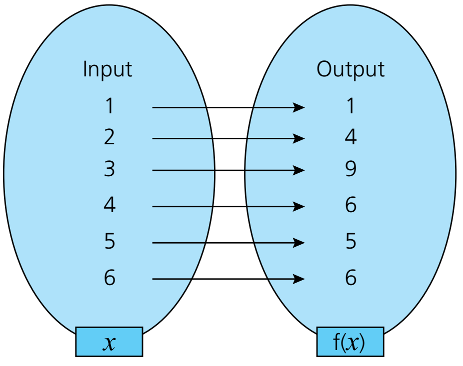
Discussion point

Look at the display on this fuel pump. One of the quantities is measured and one is calculated from it. Which is which?

You can think of the display as a mapping. Some of the values are shown below.

Amount of fuel (litres)		Cost ($)
1	→	2.40
2	→	4.80
3	→	7.20
4	→	9.60
5	→	12.00
10	→	24.00
50	→	120.00
100	→	240.00

If x is an element of the first set, then $f(x)$ denotes the associated element from the second set. For example, this mapping diagram shows integers mapped onto the final digit of their squares.



Discussion point

Which digits will never appear in the output set of the previous example?

A **function** is a rule that associates each element of one set (the **input**) with only one element of a second set (the **output**). It is possible for more than one input to have the same output, as shown above.

You can use a **flow chart** (or **number machine**) to express a function.

This flow chart shows a function, f, with two operations. The first operation is ×2 and the second operation is +3.

Input ⟶ [× 2] ⟶ [+ 3] ⟶ Output

You can write the equation of a line in the form $y = 2x + 3$ using **function notation**.

Read this as 'f of x equals two x plus three'.

$$f(x) = 2x + 3$$

or

$$f: x \mapsto 2x + 3$$

Read this as 'f maps x onto two x plus three'.

Using this notation, you can write, for example:

$$f(4) = 2 \times 4 + 3 = 11$$

or

$$f:(-5) \mapsto 2 \times (-5) + 3 = -7$$

The domain and range

The **domain** of a function f(x) is the set of all possible inputs. This is the set of values of x that the function operates on. In the first mapping diagram of the next worked example, the domain is the first five positive odd numbers. If no domain is given, it is assumed to be all real values of x. This is often denoted by the letter $\mathbb{R}$.

Real numbers are all of the rational and irrational numbers.

The **range** of the function f(x) is all the possible output values, i.e. the corresponding values of f(x). It is sometimes called the **image set** and is controlled by the domain.

In certain functions one or more values must be excluded from the domain, as shown in the following example.

Worked example

For the function $f(x) = \frac{1}{2x+1}$:

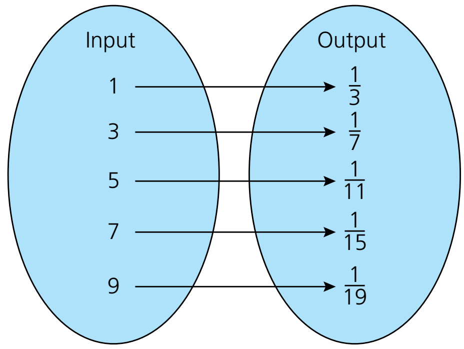
a Draw a mapping diagram showing the outputs for the set of inputs odd numbers from 1 to 9 inclusive.

b Draw a mapping diagram showing the outputs for the set of inputs even numbers from 2 to 10 inclusive.

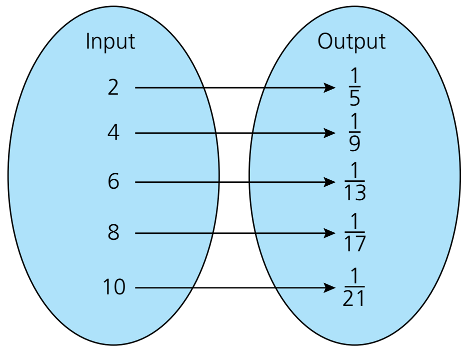
c Which number cannot be an input for this function?

Solution

a



b



c A fraction cannot have a denominator of 0, so $2x + 1 \neq 0$

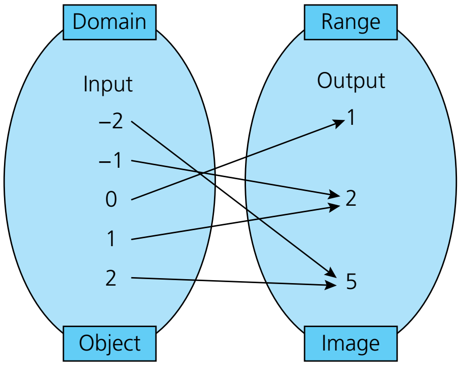
$\Rightarrow x = -\frac{1}{2}$ must be excluded.

Mappings

A mapping is the process of going from an object to its image.

For example, this mapping diagram shows the function $f(x) = x^2 + 1$ when the domain is the set of integers $-2 \leqslant x \leqslant 2$.

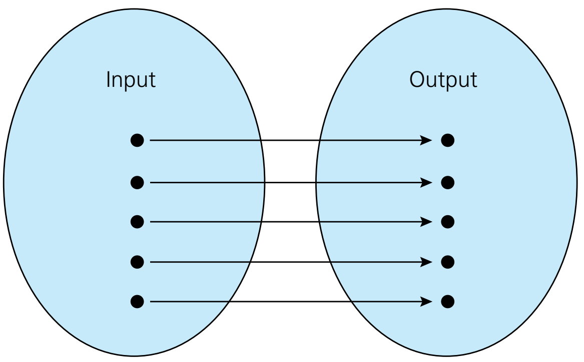
A mapping diagram is one way to illustrate a function.

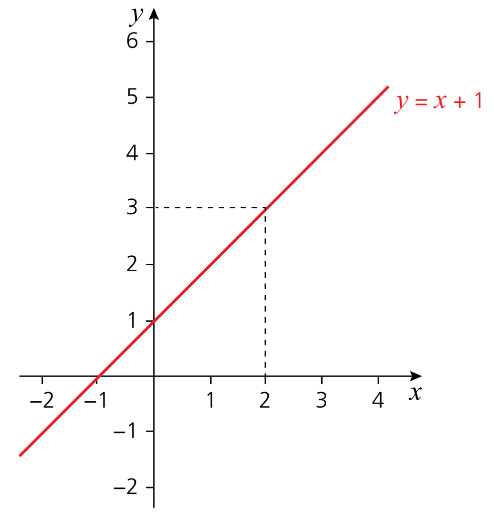


There are four different types of mappings.

One-one

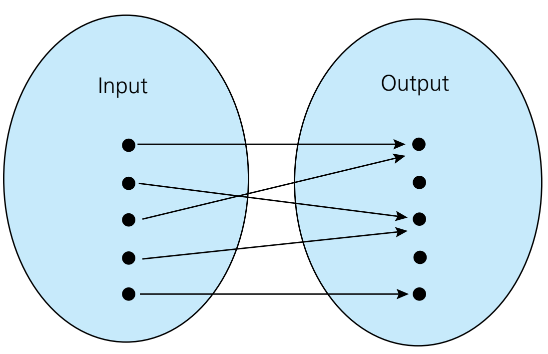
Every object has a unique image and every image comes from only one object.

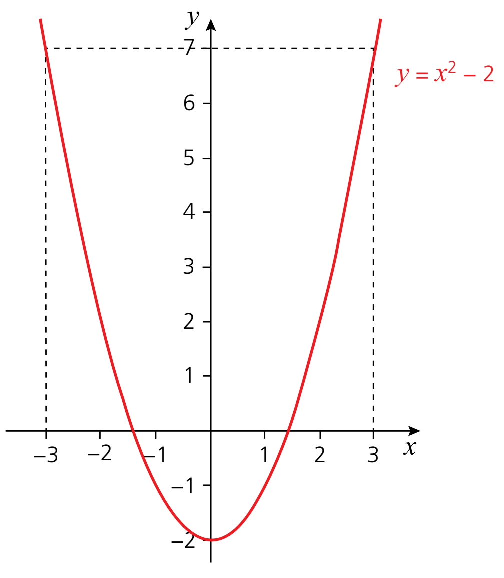




Many-one

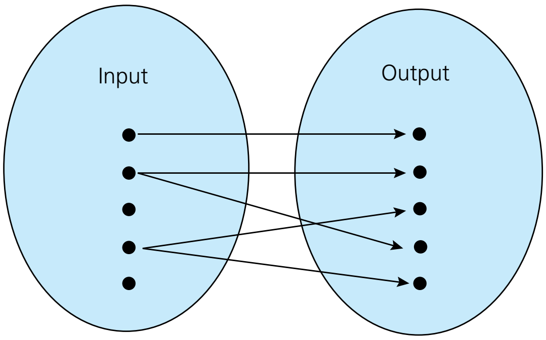
Every object has a unique image but at least one image corresponds to more than one object.

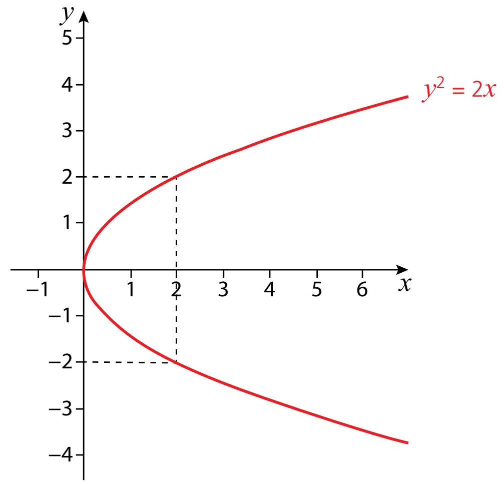




One-many

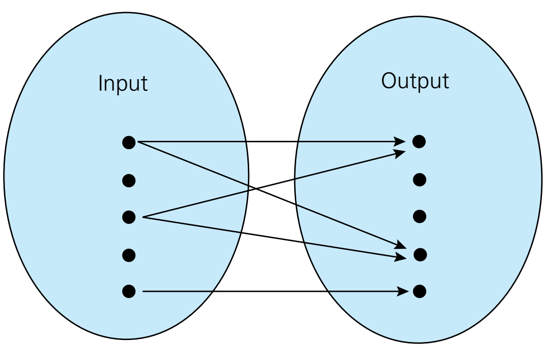
There is at least one object that has more than one image but every image comes from only one object.

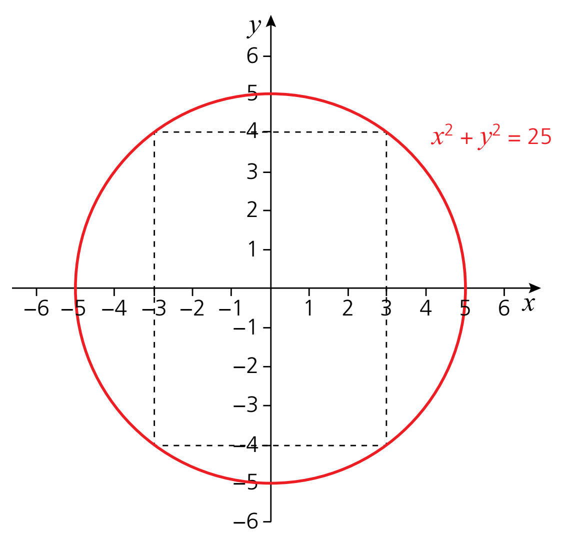




Many-many

There is at least one object that has more than one image and at least one image that corresponds to more than one object.





Types of function

A function is a mapping that is either one-one or many-one.

For a one-one function, the graph of y against x doesn't 'double back' on itself.

Below are some examples of one-one functions.

- All straight lines that are not parallel to either axis.
- Functions of the form $y = x^{2n+1}$ for integer values of n.
- Functions of the form $y = a^x$ for $a > 0$.
- $y = \cos x$ for $0° \leqslant x \leqslant 180°$.

These are examples of many-one functions:

- all quadratic curves,
- cubic equations with two turning points.

Worked example

Sketch each function and state whether it is one-one or many-one.

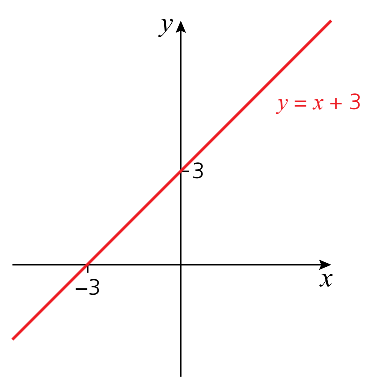
a $y = x + 3$ **b** $y = x^2 - 1$

Solution

a $y = x + 3$ is a straight line.

When $x = 0$, $y = 3$, so the point $(0, 3)$ is on the line.

When $y = 0$, $x = -3$, so the point $(-3, 0)$ is on the line.



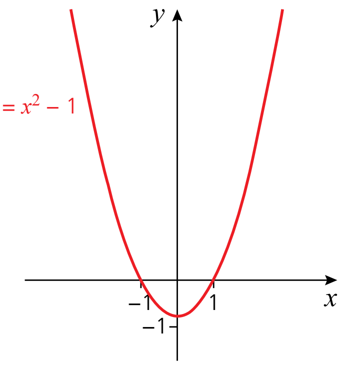
$y = x + 3$ is a one-one function.

b $y = x^2$ is a ∪-shaped curve through the origin.

$y = x^2 - 1$ is the same shape, but has been moved down one unit so crosses the y-axis at $(0, -1)$.

$y = x^2 - 1$ factorises to $y = (x + 1)(x - 1)$

$\Rightarrow$ When $y = 0$, $x = 1$ or $x = -1$.



$y = x^2 - 1$ is a many-one function since, for example, $y = 0$ corresponds to both $x = 1$ and $x = -1$.

Inverse function

The inverse function reverses the effect of the function. For example, if the function says 'double', the inverse says 'halve'; if the function says 'add 2', the inverse says 'subtract 2'. All one-one functions have an inverse; many-one functions do not.

Worked example

a Use a flow chart to find the inverse of the function $f(x) = \frac{3x+2}{2}$.

b Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes. Use the same scale on both axes.

c What do you notice?

Solution

a For $f(x) = \frac{3x+2}{2}$:

Input $\longrightarrow$ $\boxed{\times 3}$ $\longrightarrow$ $\boxed{+2}$ $\longrightarrow$ $\boxed{\div 2}$ $\longrightarrow$ Output

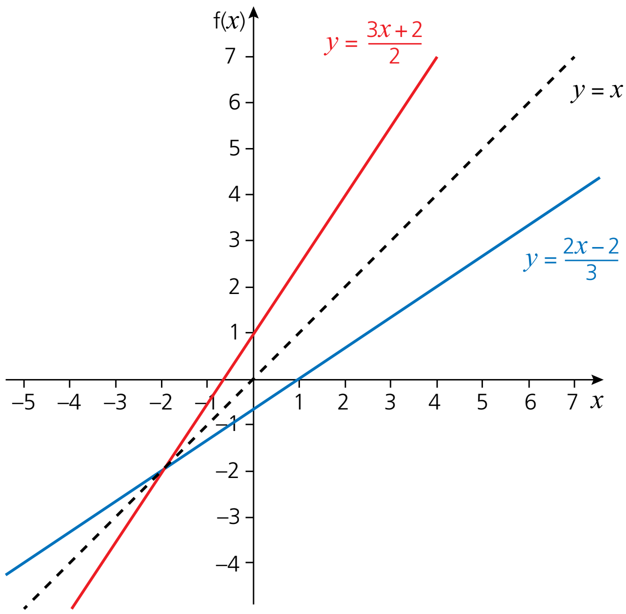
$x \longrightarrow 3x \longrightarrow 3x + 2 \longrightarrow \frac{3x+2}{2} \longrightarrow f(x)$

Reversing these operations gives the inverse function.

Output $\longleftarrow$ $\boxed{\div 3}$ $\longleftarrow$ $\boxed{-2}$ $\longleftarrow$ $\boxed{\times 2}$ $\longleftarrow$ Input

$f^{-1}(x) \longleftarrow \frac{2x-2}{3} \longleftarrow 2x - 2 \longleftarrow 2x \longleftarrow x$

b



Reflecting in the line $y = x$ has the effect of switching the x- and y-coordinates.

c The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections of each other in the line $y = x$.

An alternative method is to interchange the coordinates, since this gives a reflection in the line $y = x$, and then use an algebraic method to find the inverse as shown in the next example.

Worked example

a Find $\mathrm{g}^{-1}(x)$ when $\mathrm{g}(x) = \frac{x}{3} + 4$.

b Sketch $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$ on the same axes. Use the same scale on both axes.

Solution

a Let $y = \frac{x}{3} + 4$.

Interchange x and y. $\quad x = \frac{y}{3} + 4$

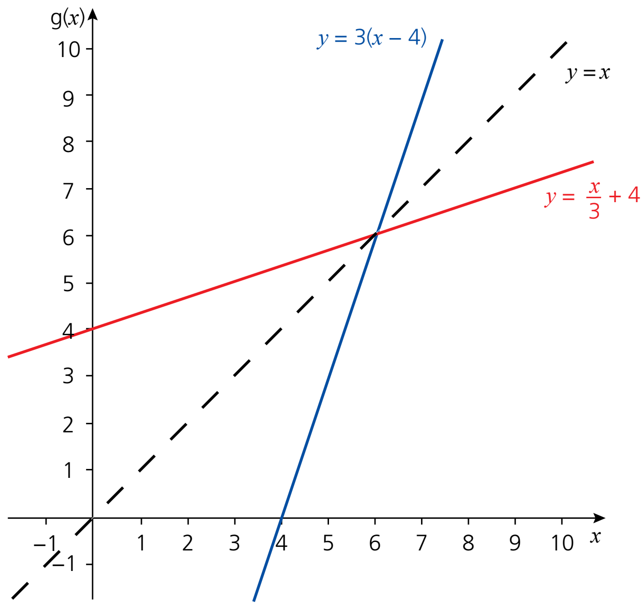
Rearrange to make y the subject. $\quad x - 4 = \frac{y}{3}$

$\Rightarrow y = 3(x - 4)$

The inverse function is given by $\mathrm{g}^{-1}(x) = 3(x - 4)$.

Rearranging and interchanging x and y can be done in either order.

b



Worked example

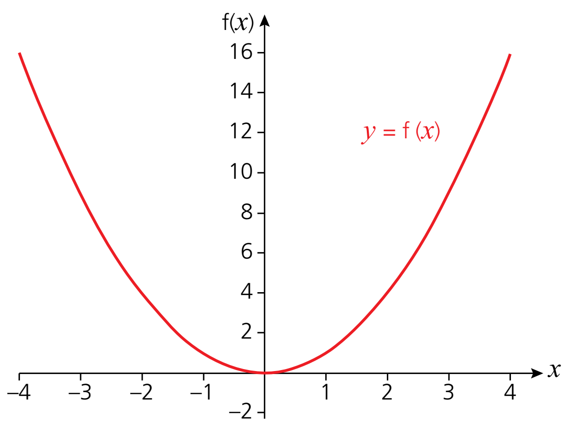
a Sketch the graph of the function $f(x) = x^2$ for $-4 \leqslant x \leqslant 4$.

b Explain, using an example, why $f(x)$ does not have an inverse with $-4 \leqslant x \leqslant 4$ as its domain.

c Suggest a suitable domain for $f(x)$ so that an inverse can be found.

Solution

a



b The function does not have an inverse with $-4 \leqslant x \leqslant 4$ as its domain because, for example, f(2) and f(–2) both equal 4. This means that if the function were reversed, there would be no unique value for 4 to return to. In other words, $f(x) = x^2$ is not a one-one function for $-4 \leqslant x \leqslant 4$.

c Any domain in which the function is one-one, for example, $0 \leqslant x \leqslant 4$.

Exercise 1.1

1 For the function $f(x) = 3x + 4$, find:

a $f(3)$ b $f(-2)$ c $f(0)$ d $f\left(\frac{1}{2}\right)$

2 For the function $g(x) = (x + 2)^2$, find:

a $g(4)$ b $g(-6)$ c $g(0)$ d $g\left(\frac{1}{2}\right)$

3 For the function $h: x \to 3x^2 + 1$, find:

a $h(2)$ b $h(-3)$ c $h(0)$ d $h\left(\frac{1}{3}\right)$

4 For the function $f: x \to \frac{2x+6}{3}$, find:

a $f(3)$ b $f(-6)$ c $f(0)$ d $f\left(\frac{1}{4}\right)$

$\sqrt{2x+1}$ is the notation for the positive square root of $2x+1$

5 For the function $f(x) = \sqrt{2x+1}$:

a Draw a mapping diagram to show the outputs when the set of inputs is the odd numbers from 1 to 9 inclusive.
b Draw a mapping diagram to show the outputs when the set of inputs is the even numbers from 2 to 10 inclusive.
c Which number must be excluded as an input?

6 Find the range of each function:

a $f(x) = 3x - 2$; domain $\{1, 2, 3, 4, 5\}$
b $g(x) = \frac{x-4}{2}$; domain $\{-2, -1, 0, 1, 2\}$
c $h(x) = 2x^2$; domain $x \in \mathbb{R}$
d $f: x \to x^2 + 6$; domain $x \in \mathbb{R}$

7 Which value(s) must be excluded from the domain of these functions?

a $f(x) = \frac{1}{x}$ b $f(x) = \sqrt{x-1}$
c $f(x) = \frac{3}{2x-3}$ d $f(x) = \sqrt{2-x^2}$

8 Find the inverse of each function:

a $f(x) = 7x - 2$ b $g(x) = \frac{3x+4}{2}$
c $h(x) = (x - 1)^2$ for $x \geqslant 1$ d $f(x) = x^2 + 4$ for $x \geqslant 0$

9 a Find the inverse of the function $f(x) = 3x - 4$.
b Sketch $f(x)$, $f^{-1}(x)$ and the line $y = x$ on the same axes. Use the same scale on both axes.

Plot: Start with a table of values. Sketch: Show the main features of the curve.

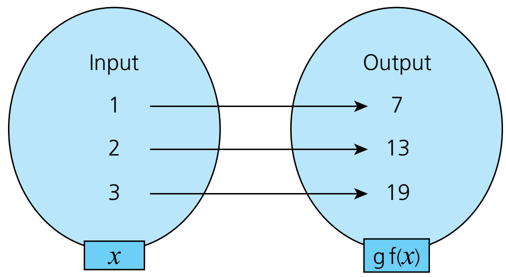
10 a Plot the graph of the function $f(x) = 4 - x^2$ for values of x such that $0 \leqslant x \leqslant 3$. Use the same scale on both axes.
b Find the values of $f^{-1}(-5)$, $f^{-1}(0)$, $f^{-1}(3)$ and $f^{-1}(4)$.
c Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same axes. Use the range -6 to $+6$ for both axes.

Composition of functions

When two functions are used one after the other, the single equivalent function is called the **composite function**.

For example, if $f(x) = 3x + 2$ and $g(x) = 2x - 3$, then the composite function $gf(x)$ is obtained by applying f first and then applying g to the result.

Input	→	$f(x)$	→	$gf(x)$	→	Output
x	→	$3x + 2$	→	$2(3x + 2) - 3$	→	$6x + 1$



If $f(x)$ and $g(x)$ are two functions such that the range of f is the domain of g, then $gf(x) = g(f(x))$ means that you apply f first and then apply g to the result.

$f^2(x)$ is the same as $f(f(x))$ and means that you apply the same function twice.

The order in which these operations are applied is important, as shown below.

Worked example

Given that $f(x) = 2x$, $g(x) = x^2$ and $h(x) = \frac{1}{x}$, find:

a $fg(x)$ **b** $gf(x)$ **c** $h^2(x)$

d $fgh(x)$ **e** $hgf(x)$

Solution

a $fg(x) = f(x^2)$
$= 2x^2$

b $gf(x) = g(2x)$
$= (2x)^2$
$= 4x^2$

c $h^2(x) = h[h(x)]$
$= h\left(\frac{1}{x}\right)$
$= 1 \div \frac{1}{x}$
$= x$

d $fgh(x) = fg\left(\frac{1}{x}\right)$
$= f\left[\left(\frac{1}{x}\right)^2\right]$
$= f\left(\frac{1}{x^2}\right)$
$= \frac{2}{x^2}$

e $hgf(x) = hg(2x)$
$= h((2x)^2)$
$= h(4x^2)$
$= \frac{1}{4x^2}$

Worked example

a Find $f^{-1}(x)$ when $f(x) = \frac{2x-1}{4}$

b Find $f[f^{-1}(x)]$.

c Find $f^{-1}[f(x)]$.

d What do you notice?

Solution

a Write $f(x)$ as $y = \frac{2x-1}{4}$

Interchange x and y. $x = \frac{2y-1}{4}$

$$\Rightarrow 4x = 2y - 1$$

$$\Rightarrow 2y = 4x + 1$$

$$\Rightarrow y = \frac{4x+1}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{4x+1}{2}$$

b $f[f^{-1}(x)] = f\left[\frac{4x+1}{2}\right]$

$$= \frac{2\left(\frac{4x+1}{2}\right)-1}{4}$$

$$= \frac{(4x+1)-1}{4}$$

$$= \frac{4x}{4}$$

$$= x$$

c $f^{-1}[f(x)] = f^{-1}\left(\frac{2x-1}{4}\right)$

$$= \frac{4\left(\frac{2x-1}{4}\right)+1}{2}$$

$$= \frac{(2x-1)+1}{2}$$

$$= \frac{2x}{2}$$

$$= x$$

This result is true for all functions that have an inverse.

d Questions **a** and **b** show that $f[f^{-1}(x)] = f^{-1}[f(x)] = x$.

The examples above show that applying a function and its inverse in either order leaves the original quantity unchanged, which is what the notation $f(f^{-1})$ or $f^{-1}(f)$ implies.

Worked example

Using the functions $f(x) = \sin x$ and $g(x) = x^2$, express the following as functions of x:

a $fg(x)$ **b** $gf(x)$ **c** $f^2(x)$

Solution

a $\text{fg}(x) = \text{f}[\text{g}(x)]$
$= \sin(x^2)$

b $\text{gf}(x) = \text{g}[\text{f}(x)]$
$= (\sin x)^2$

c $\text{f}^2(x) = \text{f}[\text{f}(x)]$
$= \sin(\sin x)$

Notice that $\sin(x^2)$ is not the same as $(\sin x)^2$ or $\sin(\sin x)$.

The modulus function

The **modulus** of a number is its positive value even when the number itself is negative.

The modulus is denoted by a vertical line on each side of the number and is sometimes called the **magnitude** of the quantity.

For example, $|28| = 28$ and $|-28| = 28$

$|x| = x$ when $x \geqslant 0$ and $|x| = -x$ when $x < 0$

Therefore for the graph of the modulus function $y = |\text{f}(x)|$, any part of the corresponding graph of $y = \text{f}(x)$ where $y < 0$, is reflected in the x-axis.

Worked example

For each of the following, sketch $y = \text{f}(x)$ and $y = |\text{f}(x)|$ on separate axes:

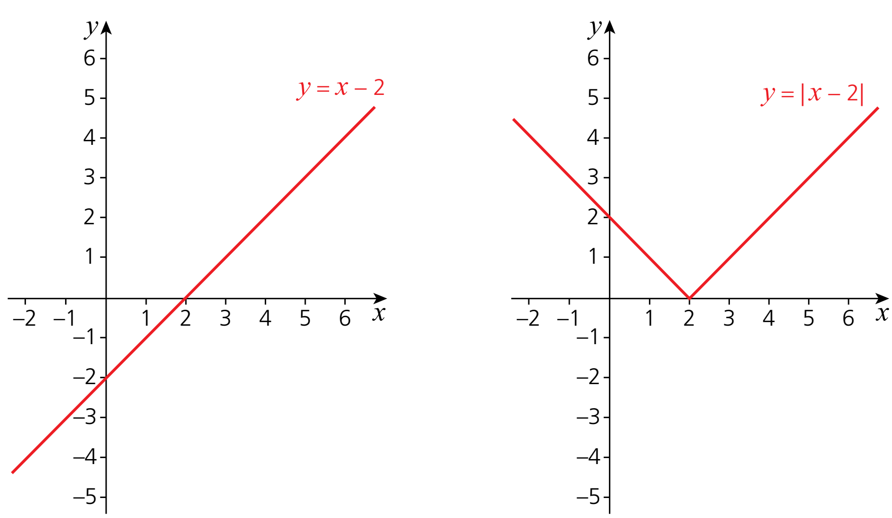
a $y = x - 2; \quad -2 \leqslant x \leqslant 6$

b $y = x^2 - 2; \quad -3 \leqslant x \leqslant 3$

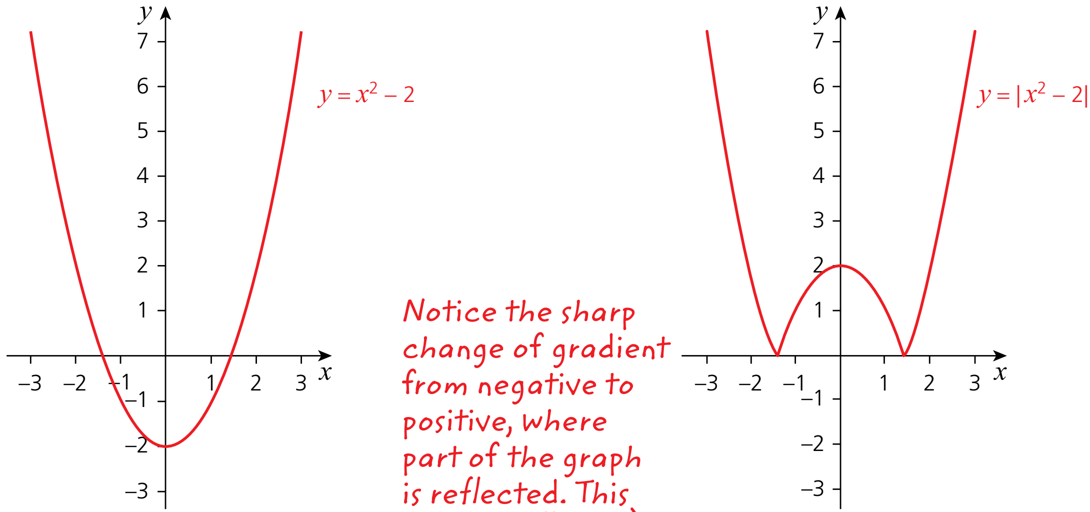
c $y = \cos x; \quad 0° \leqslant x \leqslant 180°$

Solution

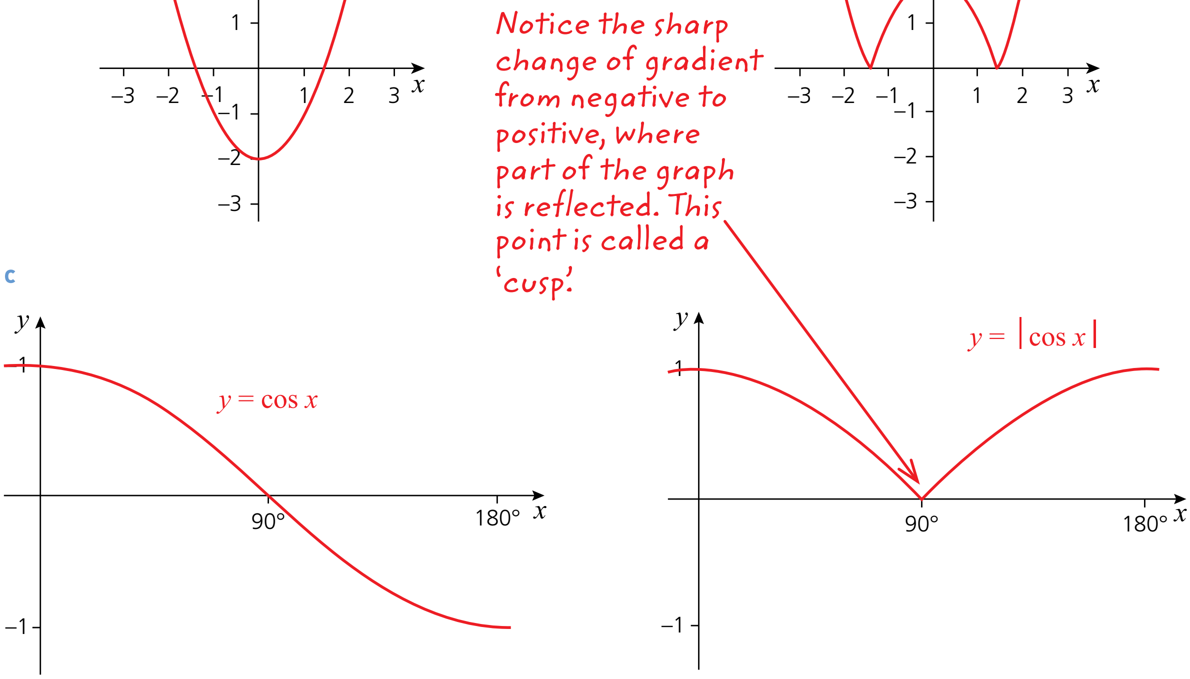
a



b



c



Exercise 1.2

1 Given that $f(x) = 3x + 2$, $g(x) = x^2$ and $h(x) = 2x$, find:

a $fg(2)$ **b** $fg(x)$ **c** $gh(x)$ **d** $fgh(x)$

2 Given that $f(x) = \sqrt{2x+1}$ and $g(x) = 4 - x$, find:

a $fg(-4)$ **b** $gf(12)$ **c** $fg(x)$ **d** $gf(x)$

3 Given that $f(x) = x + 4$, $g(x) = 2x^2$ and $h(x) = \frac{1}{2x+1}$, find:

a $f^2(x)$ **b** $g^2(x)$ **c** $h^2(x)$ **d** $hgf(x)$

4 For each function, find the inverse and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes. Use the same scale on both axes.

a $f(x) = 3x - 1$ **b** $f(x) = x^3, x > 0$

5 Solve the following equations:

a $|x - 3| = 4$ **b** $|2x + 1| = 7$ **c** $|3x - 2| = 5$ **d** $|x + 2| = 2$

6 Sketch the graph of each function:

a $y = x + 2$ **b** $y = |x + 2|$ **c** $y = |x + 2| + 3$

7 Sketch these graphs for $0° \leqslant x \leqslant 360°$:

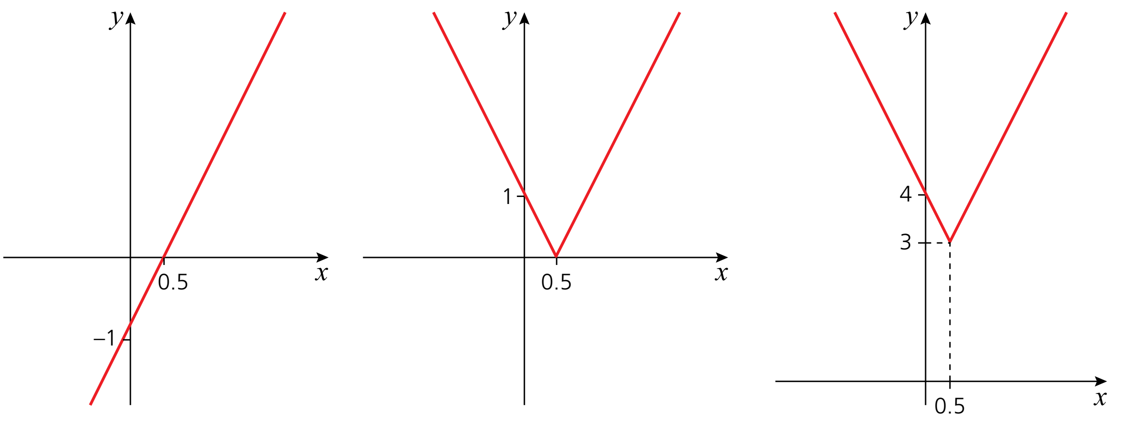
a $y = \cos x$ **b** $y = \cos x + 1$

c $y = |\cos x|$ **d** $y = |\cos x| + 1$

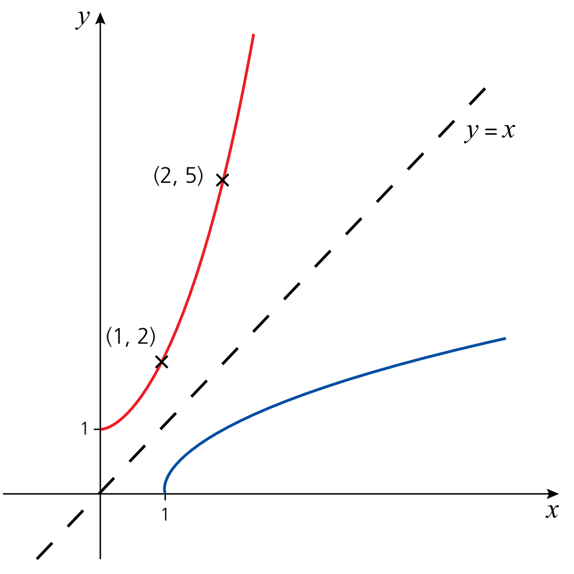
8 Graph 1 represents the line $y = 2x - 1$. Graph 2 is related to Graph 1 and Graph 3 is related to Graph 2.

Write down the equations of Graph 2 and Graph 3.

Graph 1 Graph 2 Graph 3



9 The graph shows part of a quadratic curve and its inverse.



a What is the equation of the curve?

b What is the equation of the inverse?

10 a Sketch the graphs of these functions:

i $y = 1 - 2x$ **ii** $y = |1 - 2x|$

iii $y = -|1 - 2x|$ **iv** $y = 3 - |1 - 2x|$

b Use a series of transformations to sketch the graph of $y = |3x + 1| - 2$.

Exercise 1.2 (cont)

11 For each part:

a Sketch both graphs on the same axes.

b Write down the coordinates of their points of intersection.

i $y = |x|$ and $y = 1 - |x|$

ii $y = 2|x|$ and $y = 2 - |x|$

iii $y = 3|x|$ and $y = 3 - |x|$

Past-paper questions

1 The functions f and g are defined by

$$f(x) = \frac{2x}{x+1} \text{ for } x > 0,$$

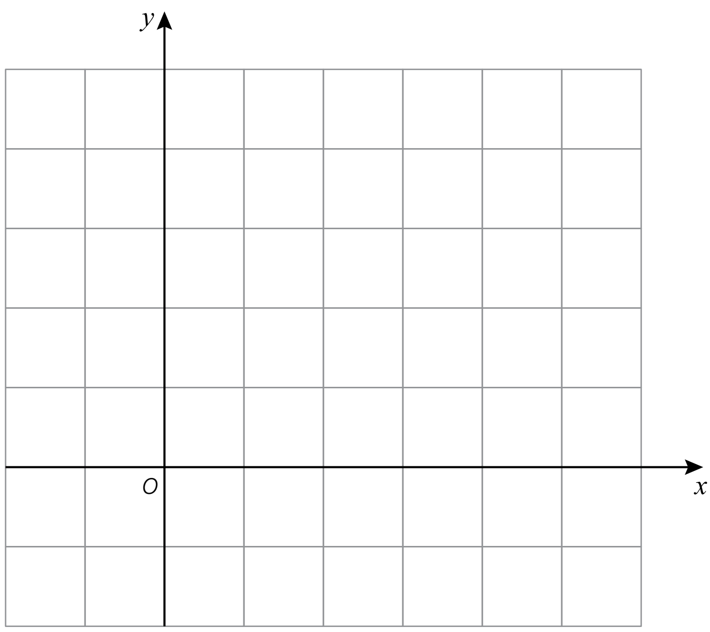
$$g(x) = \sqrt{x+1} \text{ for } x > -1$$

(i) Find fg(8). [2]

(ii) Find an expression for $f^2(x)$, giving your answer in the form $\frac{ax}{bx+c}$, where a, b and c are integers to be found. [3]

(iii) Find an expression for $g^{-1}(x)$, stating its domain and range. [4]

(iv) On axes like the ones below, sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$, indicating the geometrical relationship between the graphs. [3]



Cambridge O Level Additional Mathematics 4037
Paper 21 Q12 June 2014
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q12 June 2014

2 (i) Sketch the graph of $y = |3x - 5|$, for $-2 \leqslant x \leqslant 3$, showing the coordinates of the points where the graph meets the axes. [3]

(ii) On the same diagram, sketch the graph of $y = 8x$. [1]

(iii) Solve the equation $8x = |3x - 5|$. [3]

Cambridge O Level Additional Mathematics 4037
Paper 13 Q7 November 2010
Cambridge IGCSE Additional Mathematics 0606
Paper 13 Q7 November 2010

Learning outcomes

Now you should be able to:

- ★ understand the terms function, domain, range (image set), one-one function, inverse function and composite function
- ★ use the notation $f(x) = \sin x$, $f: x \mapsto g(x)$, $x > 0$, $f^{-1}(x)$ and $f^2(x)$ $[= f(f(x))]$
- ★ understand the relationship between $y = f(x)$ and $y = |f(x)|$, where $f(x)$ may be linear, quadratic or trigonometric
- ★ explain in words why a given function is a function or why it does not have an inverse
- ★ find the inverse of a one-one function and form composite functions
- ★ use sketch graphs to show the relationship between a function and its inverse.

Key points

- A **mapping** is a rule for changing one number into another number or numbers.
- A **function**, $f(x)$, is a rule that maps one number onto another single number.
- The **graph of a function** has only one value of y for each value of x. However, two or more values of x may give the same value of y.
- A **flow chart** can be used to show the individual operations within a function in the order in which they are applied.
- The **domain** of a function is the set of **input values**, or **objects**, that the function is operating on.
- The **range** or **image set** of a function is the corresponding set of **output values** or **images**, $f(x)$.
- A **mapping diagram** can be used to illustrate a function. It is best used when the domain contains only a small number of values.
- In a **one-one function** there is a unique value of y for every value of x and a unique value of x for every value of y.
- In a **many-one** function two or more values of x correspond to the same value of y.
- In a **one-many** function one value of x corresponds to two or more values of y.
- In a **many-many** function two or more values of x correspond to the same value of y and two or more values of y correspond to the same value of x.
- The **inverse** of a function reverses the effect of the function. Only one-one functions have inverses.
- The term **composition of functions** is used to describe the application of one function followed by another function(s).The notation $fg(x)$ means that the function g is applied first, then f is applied to the result.
- The **modulus** of a number or a function is always a positive value. $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x < 0$.
- The modulus of a function $y = f(x)$ is denoted by $|f(x)|$ and is illustrated by reflecting any part of the graph where $y < 0$ in the x-axis.