9 Circular measure

A circle is the reflection of eternity. It has no beginning and no end.
Maynard James Keenan (1964 –)

Discussion point

This is the Singapore Flyer. It has a radius of 75 metres. It takes about 30 minutes to complete one rotation, travelling at a constant speed. How fast do the capsules travel?

The tradition of measuring angles in degrees, and there being 360 degrees in one revolution, is thought to have come about because much of early mathematics was connected to astronomy, and the shepherd-astronomers of Sumeria believed that there were 360 days in a year.

The following notation is used in this chapter:

C represents the **circumference** of the circle – the distance round the circle.

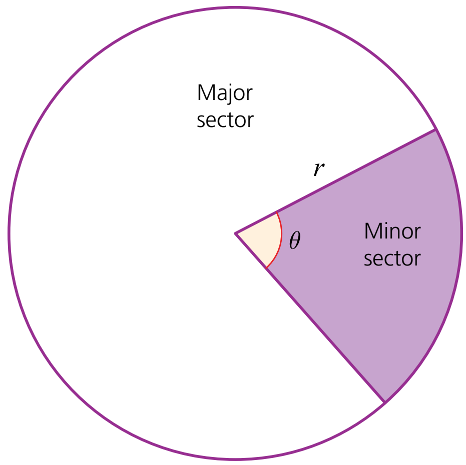
r represents the **radius** of the circle – the distance from the centre to any point on the circumference.

θ (the Greek letter theta) is used to represent the **angle** that an arc subtends at the centre of the circle.

A represents **area** – this may be the area of a whole circle or a sector.

Arc length and area of a sector

A **sector** of a circle looks similar to a piece of cake – it is the shape enclosed by an arc of the circle and two radii. If the angle at the centre is less than 180° it is called a **minor sector**, and if it is between 180° and 360° it is called a **major sector.**



Using ratios:

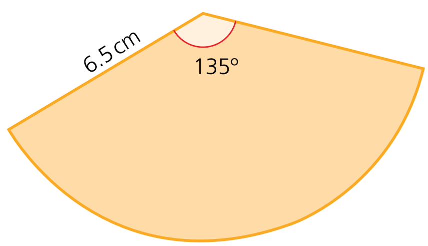
$$\frac{\text{arc length}}{\text{circumference of the circle}} = \frac{\text{area of the sector}}{\text{area of the circle}} = \frac{\theta}{360}$$

Worked example

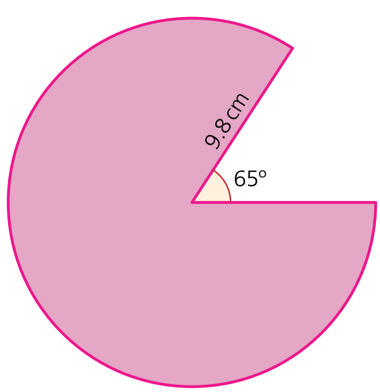
For each sector, calculate:

i the arc length ii the area iii the perimeter.

a



b



Solution

a **i** $\frac{\text{arc length}}{2\pi r} = \frac{\theta}{360}$ → arc length $= \frac{135}{360} \times 2 \times \pi \times 6.5$

$= 15.3\text{ cm (3 s.f.)}$

ii $\frac{\text{area}}{\pi r^2} = \frac{\theta}{360}$ → area $= \frac{135}{360} \times \pi \times 6.5^2$

$= 49.8\text{ cm}^2\text{ (3 s.f.)}$

iii perimeter = arc length + 2 × radius

$= 15.3 + 2(6.5) = 28.3\text{ cm (3 s.f.)}$

b The angle of this sector is 360 – 65 = 295°

i $\frac{\text{arc length}}{2\pi r} = \frac{\theta}{360}$ → arc length $= \frac{295}{360} \times 2 \times \pi \times 9.8$

$= 50.5\text{ cm (3 s.f.)}$

ii $\frac{\text{area}}{\pi r^2} = \frac{\theta}{360}$ → area $= \frac{295}{360} \times \pi \times 9.8^2$

$= 247\text{ cm}^2\text{ (3 s.f.)}$

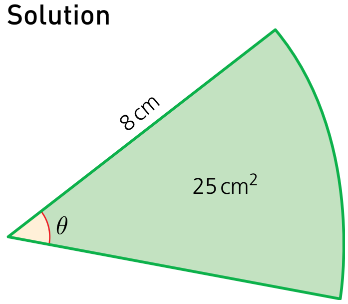
iii perimeter = arc length + 2 × radius

$= 50.5 + 2(9.8) = 70.1\text{ cm (3 s.f.)}$

Worked example

A sector of a circle of radius 8 cm has an area of 25 cm^2. Work out the angle at the centre.

Solution



Using sector area $= \frac{x}{360} \times \pi r^2$

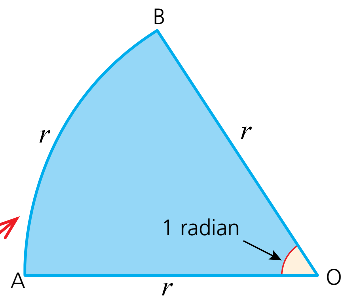
$$25 = \frac{\theta}{360} \times \pi \times 8^2$$

$$\Rightarrow \theta = \frac{25 \times 360}{\pi \times 64}$$

$$= 44.8° \text{ (3 s.f.)}$$

Radian measure

Radian measure is used extensively in mathematics because it simplifies many angle calculations. One radian (rad) is the angle in a sector when the arc length is equal to the radius. 1 rad is approximately 57.3°.



1 radian can also be written as 1^c.

Since the circumference of a circle is of length $2\pi r$, there are 2π arcs of length r round the circumference. This means that there are 2π radians in 360°.

Degrees	Radians
360	2π
180	π
90	$\frac{\pi}{2}$
60	$\frac{\pi}{3}$
45	$\frac{\pi}{4}$
30	$\frac{\pi}{6}$

1 degree is the same as $\frac{\pi}{180}$ radians, therefore:

- multiply by $\frac{\pi}{180}$ to convert degrees to radians
- multiply by $\frac{180}{\pi}$ to convert radians to degrees.

Note

- An angle given as a fraction of π is assumed to be in radians.
- If an angle is a simple fraction of 180°, its equivalent value in radians is usually expressed as a fraction of π.

Worked example

a Express the following in radians: **i** 75° **ii** 49°

b Express the following in degrees: **i** $\frac{\pi}{10}$ radians **ii** 1.25 radians

Solution

a **i** $75° = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$ radians

ii $49° = 49 \times \frac{\pi}{180} = 0.855$ radians (3 s.f.)

b **i** $\frac{\pi}{10}$ radians $= \frac{\pi}{10} \times \frac{180}{\pi} = 18°$

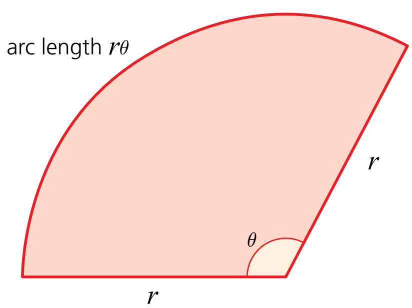
ii 1.25 radians $= 1.25 \times \frac{180}{\pi} = 71.6°$ (3 s.f.)

Using your calculator

Your calculator has modes for degrees and for radians, so always make sure that it is on the correct setting for any calculations that you do. There is usually a button marked DRG for **d**egrees, **r**adians and **g**rad (you will not use grad at this stage).
To find the value of sin 2.3^c, set your calculator to the radian mode and enter sin 2.3 followed by = or EXE, depending on your calculator. You should see the value 0.74570… on your screen.

Arc length and area of a sector in radians

Using the definition of a radian, an angle of 1 radian at the centre of a circle corresponds to an arc length equal to the radius r of the circle. Therefore an angle of θ radians corresponds to an arc length of $r\theta$.



It is accepted practice to write $r\theta$, with the Greek letter at the end, rather than θr.

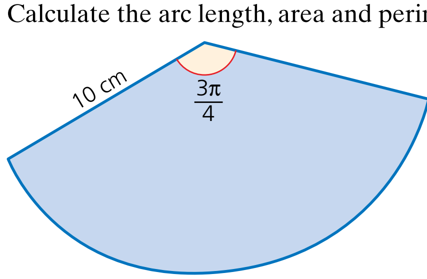
The area of this sector is the fraction $\frac{\theta}{2\pi}$ of the area of the circle (since 2π is the radian equivalent of 360°).

This gives the formula:

$$\text{area of a sector} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$

Worked example

Calculate the arc length, area and perimeter of this sector.



Solution

arc length $= 10 \times \frac{3\pi}{4}$ ← Using arc length $= r\theta$

$= \frac{15\pi}{2}$ cm

sector area $= \frac{1}{2} \times 10^2 \times \frac{3\pi}{4}$ ← Using area of sector $= \frac{1}{2}r^2\theta$

$= \frac{75\pi}{2}$ cm^2

Using perimeter = arc length + 2 × radius

perimeter $= \frac{15\pi}{2} + 2 \times 10 = \frac{15\pi}{2} + 20$ cm

Exercise 9.1

1 Express each angle in radians, leaving your answer in terms of π if appropriate:

a 120° **b** 540° **c** 22° **d** 150° **e** 37.5°

2 Express each angle in degrees, rounding your answer to 3 s.f. where necessary:

a $\frac{2\pi}{3}$ **b** $\frac{5\pi}{9}$ **c** 3^c **d** $\frac{\pi}{7}$ **e** $\frac{3\pi}{8}$

3 The table gives information about some sectors of circles.

Copy and complete the table. Leave your answers as a multiple of π where appropriate.

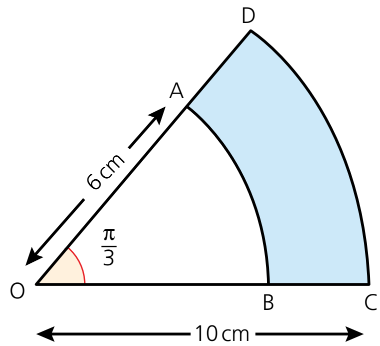
Radius, *r* (cm)	Angle at centre in degrees	Angle at centre in radians	Arc length, *s* (cm)	Area, *A* (cm^2)
8	120			
10			5	
	60		4	
6				12
	75			20

4 The table gives information about some sectors of circles. Copy and complete the table. Leave your answers as a multiple of π where appropriate.

Radius, *r* (cm)	Angle at centre in radians	Arc length, *s* (cm)	Area, *A* (cm^2)
10	$\frac{\pi}{3}$		
12		24	
	$\frac{\pi}{4}$	16	
5			25
	$\frac{3\pi}{5}$		40

Exercise 9.1 (cont)

5 OAB is a sector of a circle of radius 6 cm. ODC is a sector of a circle radius 10 cm. Angle AOB is $\frac{\pi}{3}$.

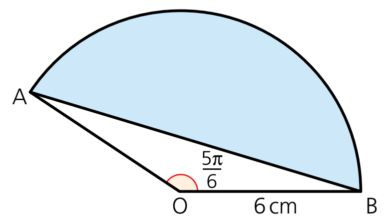


Express in terms of π:

a the area of ABCD

b the perimeter of ABCD.

6



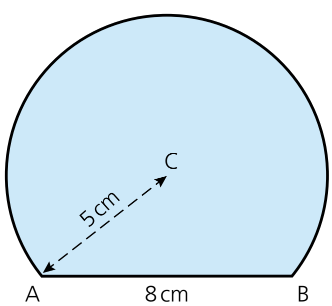
The shaded area is called a segment of a circle.

a Work out the area of the sector AOB.

b Calculate the area of the triangle AOB.

c Work out the shaded area.

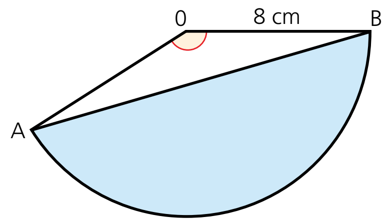
7 The diagram shows the cross-section of a paperweight. The paperweight is a sphere of radius 5 cm with the bottom cut off to create a circular flat base with diameter 8 cm.



a Calculate the angle ACB in radians.

b Work out the area of cross-section of the paperweight.

8 The perimeter of the sector in the diagram is $6\pi + 16$ cm.

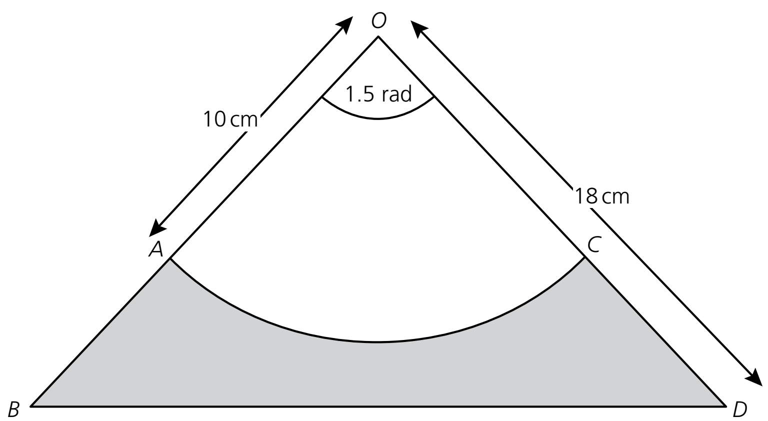


Calculate:

a angle AOB
b the exact area of sector AOB
c the exact area of the triangle AOB
d the exact area of the shaded segment.

Past–paper questions

1



The diagram shows an isosceles triangle OBD in which $OB = OD = 18$ cm and angle $BOD = 1.5$ radians. An arc of the circle, centre O and radius 10 cm, meets OB at A and OD at C.

(i) Find the area of the shaded region. [3]
(ii) Find the perimeter of the shaded region. [4]

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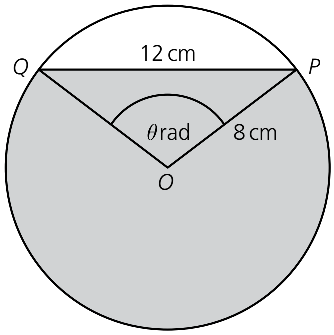
2 The diagram shows a circle, centre O, radius 8 cm. Points P and Q lie on the circle such that the chord $PQ = 12$ cm and angle $POQ = \theta$ radians.



(i) Show that $\theta = 1.696$, correct to 3 decimal places. [2]
(ii) Find the perimeter of the shaded region. [3]
(iii) Find the area of the shaded region. [3]

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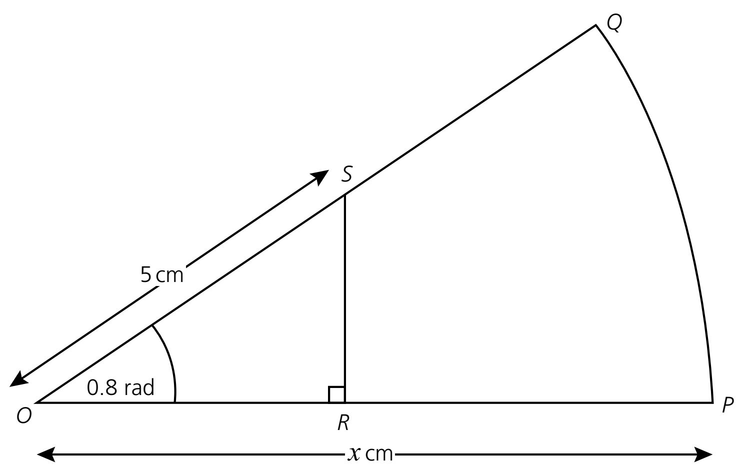
3



The diagram shows a sector OPQ of a circle with centre O and radius x cm. Angle POQ is 0.8 radians. The point S lies on OQ such that $OS = 5$ cm. The point R lies on OP such that angle ORS is a right angle. Given that the area of triangle ORS is one-fifth of the area of sector OPQ, find

(i) the area of sector OPQ in terms of x and hence show that the value of x is 8.837 correct to 4 significant figures, [5]
(ii) the perimeter of $PQSR$, [3]
(iii) the area of $PQSR$. [2]

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Learning outcomes

Now you should be able to:

★ solve problems involving the arc length and sector area of a circle, including knowledge and use of radian measure.

Key points

✔ Angles are measured either in degrees or radians.
$180° = \pi$ radians

✔ The angle at the centre of the circle subtended by an arc that is the same length as the radius is 1 radian.

✔ The formulae for area of a circle ($A = \pi r^2$) and circumference of a circle ($C = 2\pi$) are the same whether the angle is measured in degrees or radians.

✔ You will need to learn these formulae.

	Radians
Area	πr^2
Circumference	$2\pi r$
Arc length (θ at centre)	$r\theta$
Sector area (θ at centre)	$\frac{1}{2}r^2\theta$