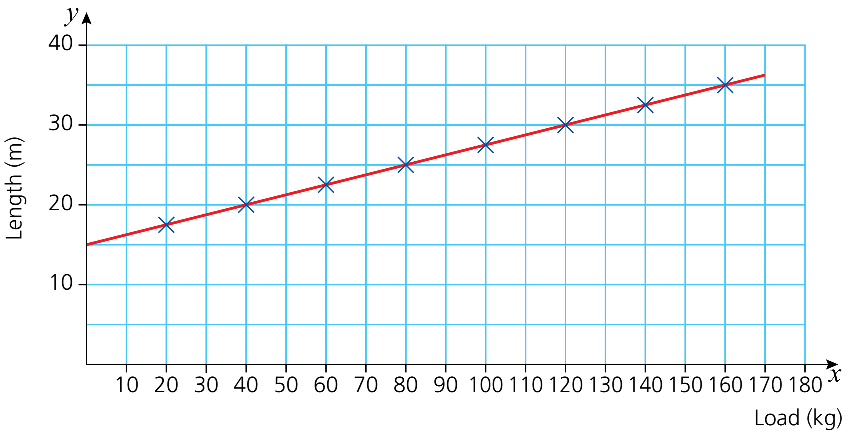
8 Straight line graphs

Every new body of discovery is mathematical in form because there is no other guidance we can have.

Charles Darwin (1809 – 1882)

Discussion point

If you do a bungee jump, you will want to be certain that the rope won't stretch too far.
In an experiment a rope is tested by hanging different loads on it and measuring its length. The measurements are plotted on this graph.



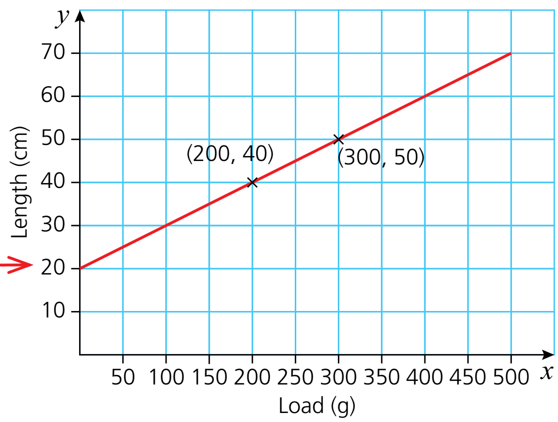
You will have met straight line graphs frequently in abstract algebraic problems, but they can also be used to find information in a practical situation such as this.

What does the graph tell you about the rope?

When a load of 200 g is attached to a spring, its stretched length is 40 cm. With a load of 300 g, its length is 50 cm. Assuming that the extension is proportional to the load, draw a graph to show the relationship between the load and the length of the spring and use it to find the natural length of the spring.

Since the load is the variable that can be directly controlled, it is plotted on the horizontal axis and the length of the spring on the vertical axis. Plotting the points (200, 40) and (300, 50) and joining them with a straight line gives the graph below.

The graph shows that the natural length of the spring, i.e. the length when there is no load attached, is 20 cm.



The information that the extension is proportional to the load tells you that the graph of the relationship will be a straight line.

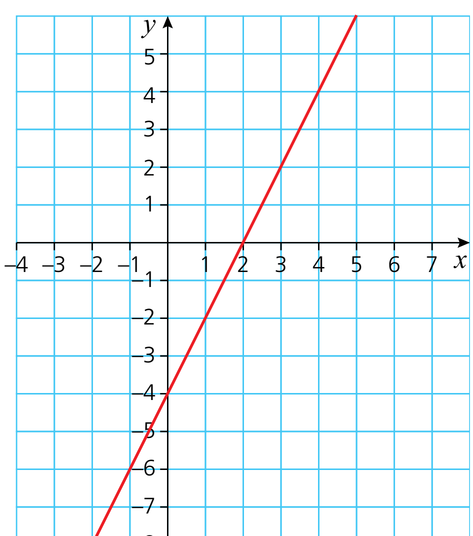
The straight line $y = mx + c$

When the equation of a straight line is written in the form $y = mx + c$, m represents the gradient of the line and the line crosses the y–axis at $(0, c)$.

You can use this to find the equation of a straight line given the graph.

Worked example

Find the equation of this straight line.



The gradient of the line joining the points (x_1, y_1) and (x_2, y_2) is given by

Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

Solution

The line crosses the y-axis at $(0,-4)$ so $c = -4$.

To find the gradient of the line, choose two points on the line and call them (x_1, y_1) and (x_2, y_2). The points of intersection with the axes, $(0,-4)$ and $(2, 0)$, are obvious choices.

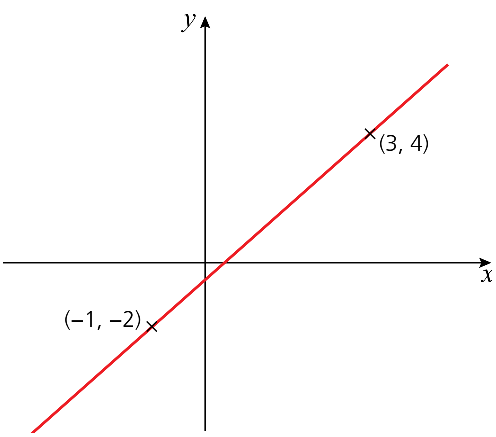
Using the gradient formula:

Gradient $(m) = \dfrac{0-(-4)}{2-0} = 2$

So the equation of the line is $y = 2x - 4$.

Worked example

Find the equation of the line shown.



Solution

Substitute each pair of coordinates into the equation $y = mx + c$.

Point $(-1,-2)$: $\quad -2 = m(-1) + c \qquad (1)$

Point $(3, 4)$: $\quad 4 = m(3) + c \qquad (2)$

Subtract equation (1) from equation (2).

$4-(-2) = (3m + c) - (-m + c)$

$\Rightarrow \quad 6 = 3m + c + m - c$

$\Rightarrow \quad 6 = 4m$

$\Rightarrow \quad m = 1.5$

Equation 2 is the more straightforward equation because it has no negative signs.

Substitute $m = 1.5$ into equation (2).

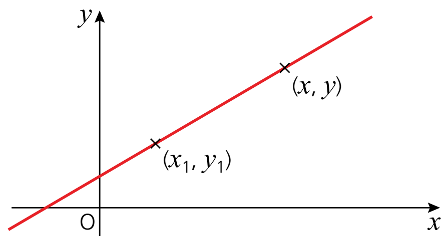
$4 = 3(1.5) + c$

$\Rightarrow \quad c = -0.5$

So the equation of the line is $y = 1.5x - 0.5$.

As well as $y = mx + c$, there are several other formulae for the equation of a straight line. One that you are likely to find useful deals with the situation where you know the gradient of the line, m, and the coordinates of one point on it, (x_1, y_1).

The equation is $y - y_1 = m(x - x_1)$.



Midpoint of a line

When a line has a fixed length, the midpoint, i.e. the point half way between the two ends of the line, has as its coordinates the average of the individual x- and y-coordinates.

Worked example

The midpoint of the line joining the points (x_1, y_1) and (x_2, y_2) is given by midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

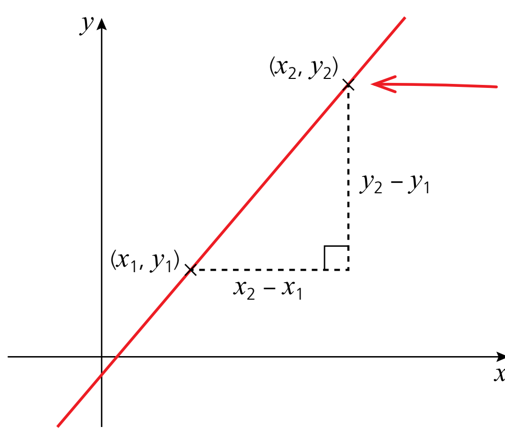
Find the midpoint of the line joining (2, 5) and (4, 13).

Solution

The coordinates of the midpoint are $\left(\frac{2+4}{2}, \frac{5+13}{2}\right) = (3, 9)$.

Length of a line

To find the length of the line joining two points, use Pythagoras' theorem.



The length of the line joining the points (x_1, y_1) and (x_2, y_2) is given by length $= \sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2)}$

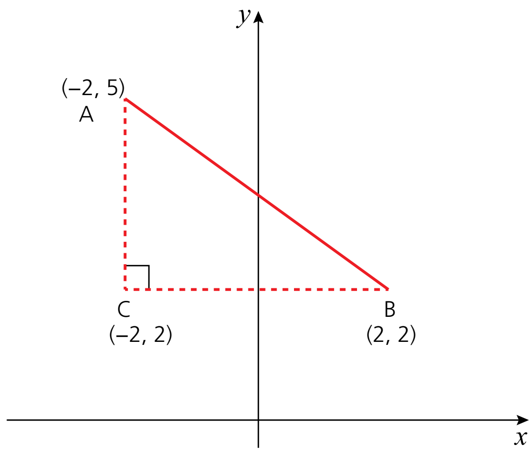
Worked example

Work out the length of the line joining the points A(–2, 5) and B(2, 2).

Solution

You can either:

- sketch the triangle and then use Pythagoras' theorem or
- use the formula given above.



From the triangle: AC = 3 units

BC = 4 units

So $AB^2 = 3^2 + 4^2$

$= 25$

AB = 5 units

Alternatively, substituting directly into the formula (without drawing a diagram) gives:

$$\text{length} = \sqrt{((2-(-2)^2 + (2-5)^2)}$$

$$= 5 \text{ units}$$

Parallel lines

Two lines are **parallel** if they have the same gradient. If you are given the equations of two straight line graphs in the form $y = mx + c$, you can immediately identify whether or not the lines are parallel. For example, $y = 3x - 7$ and $y = 3x + 2$ are parallel since they both have a gradient of 3.

If one or both of the equations are given in a different form, you will need to rearrange them in order to find out whether or not they are parallel.

Worked example

Show that the two lines $y = \frac{1}{2}x - 4$ and $x - 2y - 6 = 0$ are parallel.

Solution

Start by rearranging the second equation into the form $y = mx + c$.

$$x - 2y - 6 = 0 \Rightarrow x - 6 = 2y$$
$$\Rightarrow 2y = x - 6$$
$$\Rightarrow y = \frac{1}{2}x - 3$$

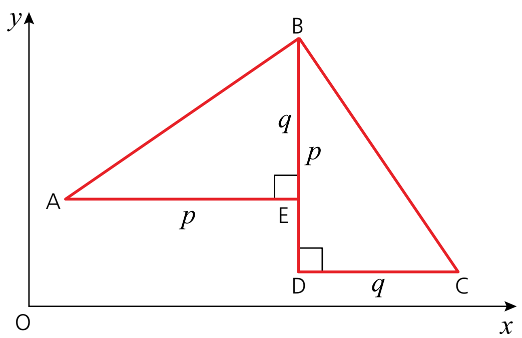
Both lines have a gradient of $\frac{1}{2}$ so are parallel.

Perpendicular lines

Two lines are **perpendicular** if they intersect at an angle of 90°.

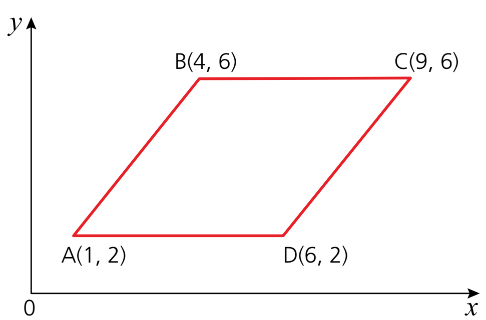
Activity

The diagram shows two congruent right-angled triangles where p and q can take any value.



1 Copy the diagram onto squared paper.
2 Explain why $\angle ABC = 90°$.
3 Calculate the gradient of AB (m_1) and the gradient of BC (m_2).
4 Show that $m_1 m_2 = -1$.

Worked example



a Explain why ABCD is a rhombus.
b Show that the diagonals AC and BD are perpendicular. (This result is always true for a rhombus.)

Solution

a A rhombus is a parallelogram with all sides equal in length.

AD and BC are both parallel to the x-axis and have length 5 units.

gradient of AB = gradient DC $= \frac{\text{increase in } y}{\text{increase in } x} = \frac{4}{3}$

$AB = DC = \sqrt{3^2 + 4^2} = 5$ units

So ABCD is a rhombus.

b Using the formula gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

gradient of AC $= \frac{6-2}{9-1} = \frac{1}{2}$

gradient of BD $= \frac{2-6}{6-4} = -2$

$\frac{1}{2} \times (-2) = -1$ so diagonals AC and BD are perpendicular.

Exercise 8.1

1 For each of the following pairs of points A and B, calculate:

i the gradient of the line AB
ii the gradient of the line perpendicular to AB
iii the length of AB
iv the coordinates of the midpoint of AB.

a A(4, 3) B(8, 11)
b A(5, 3) B(10, –8)
c A(6, 0) B(8, 15)
d A(–3, –6) B(2, –7)

2 A(0, 5), B(4, 1) and C(2, 7) are the vertices of a triangle. Show that the triangle is right angled:

a by working out the gradients of the sides
b by calculating the lengths of the sides.

3 A(3, 5), B(3, 11) and C(6, 2) are the vertices of a triangle.

a Work out the perimeter of the triangle.
b Sketch the triangle and work out its area using AB as the base.

4 A quadrilateral PQRS has vertices at P(–2, –5), Q(11, – 7), R(9, 6) and S(–4, 8).

a Work out the lengths of the four sides of PQRS.
b Find the coordinates of the midpoints of the diagonals PR and QS.
c Without drawing a diagram, show that PQRS cannot be a square. What shape is PQRS?

5 The points A, B and C have coordinates (2, 3), (6, 12) and (11, 7) respectively.

a Draw the triangle ABC.
b Show by calculation that the triangle is isosceles and write down the two equal sides.
c Work out the midpoint of the third side.
d By first calculating appropriate lengths, calculate the area of triangle ABC.

6 A triangle ABC has vertices at A(3, 2), B(4, 0) and C(8, 2).

a Show that the triangle is right angled.
b Find the coordinates of point D such that ABCD is a rectangle.

7 P(–2, 3), Q(1, q) and R(7, 0) are collinear points (i.e. they lie on the same straight line).

a Find the value of Q.
b Write down the ratio of the lengths PQ : QR.

8 A quadrilateral has vertices A(–2, 8), B(–5, 5), C(5, 3) and D(3, 7).
a Draw the quadrilateral.
b Show by calculation that it is a trapezium.
c ABCE is a parallelogram. Find the coordinates of E.

9 In each part, find the equation of the line through the given point that is:
i parallel and **ii** perpendicular to the given line.
a $y = 2x + 6$; $(5, -3)$
b $x + 3y + 5 = 0$; $(-4, 7)$
c $2x = 3y + 1$; $(-1, -6)$

10 Find the equation of the perpendicular bisector of the line joining each pair of points.
a (2, 3) and (8, –1)
b (–7, 3) and (1, 5)
c (5, 6) and (4, –3)

11 P is the point (2, –1) and Q is the point (8, 2).
a Write the equation of the straight line joining P and Q.
b Find the coordinates of M, the midpoint of PQ.
c Write the equation of the perpendicular bisector of PQ.
d Write down the coordinates of the points where the perpendicular bisector crosses the two axes.

Relationships of the form $y = ax^n$

When you draw a graph to represent a practical situation, in many cases your points will lie on a curve rather than a straight line. When the relationships are of the form $y = ax^n$ or $y = Ab^x$, you can use logarithms to convert the curved graphs into straight lines. Although you can take the logarithms to any positive base, the forms log and ln are used in most cases.

Worked example

The data in the table were obtained from an experiment. y represents the mass in grams of a substance (correct to 2 d.p.) after a time t minutes.

t	4	9	14	19	24	29
y	3.00	4.50	5.61	6.54	7.35	8.08

Saira wants to find out if these values can be modelled by the function $y = at^n$.

a By taking logarithms to the base 10 of both sides, show that the model can be written as $\log y = n \log t + \log a$.

b Explain why, if the model is valid, plotting the graph of $\log y$ against $\log t$ will result in a straight line.

c Plot the graph of $\log y$ against $\log t$ and use it to estimate the values of a and t. Hence express the relationship in the form $y = at^n$.

d Assuming that this relationship continues for at least the first hour, after how long would there be 10 g of the substance?

Solution

a $y = at^n \Rightarrow \lg y = \lg at^n$

$\Rightarrow \lg y = \lg a + \lg t^n$

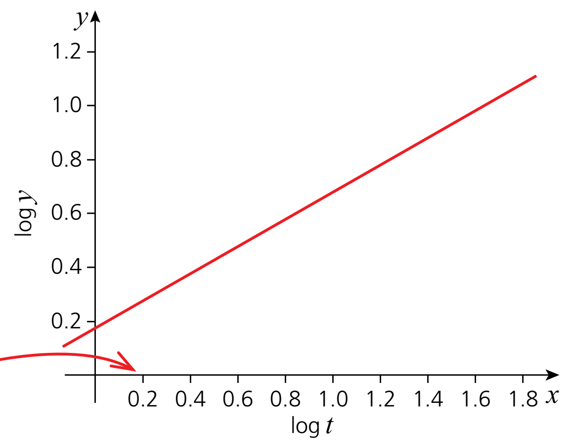
$\Rightarrow \lg y = \lg a + n\lg t$

$\Rightarrow \lg y = n\lg t + \lg a$

b Comparing this with the equation $Y = mX + c$ gives $Y = \log y$ and $X = \log t$. This shows that if the model is valid, the graph of $\log y$ (on the vertical axis) against $\log t$ will be a straight line with gradient n and intercept on the vertical axis at $\log a$.

c

t	4	9	14	19	24	29
y	3.00	4.50	5.61	6.54	7.35	8.08
log *t*	0.60	0.95	1.15	1.28	1.38	1.46
log *y*	0.48	0.65	0.75	0.82	0.87	0.91



NOTE: You cannot have a break in the horizontal axis because this would lead to an incorrect point of intersection with the y–axis.

Using the points (0, 0.18) and (1.5, 0.92), the gradient of the line is:

gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

$$\frac{0.92 - 0.18}{1.5 - 0} = 0.493$$

This is approximately equal to 0.5, so $n = 0.5$.

Since the values from the graph are only approximate, the results should only be given to 1 or 2 d.p.

Using intercept on the y–axis = lg a

$0.18 = \lg a$

$\Rightarrow a = 1.513 \approx 1.5$

Therefore the relationship is $y = 1.5t^{0.5}$ or $y = 1.5\sqrt{t}$.

Do not go beyond the values in the table unless the question tells you to. If it doesn't, you cannot be sure that the relationship you have found is valid outside of known bounds.

d There will be 10 g of the substance when $y = 10$

$\Rightarrow \quad 10 = 1.5\sqrt{t}$

$\Rightarrow 100 = 2.25t$

$\Rightarrow \quad t = \frac{100}{2.25}$

$= 44.44$

So, there will be 10 g after about 44 minutes.

Relationships of the form $y = Ab^x$

These are often referred to as exponential relationships since the variable is the power.

Worked example

The table shows the temperature, θ, recorded in degrees Celsius to the nearest degree, of a cup of coffee t minutes after it is poured and milk is added.

t	0	4	8	12	16	20
θ	80	63	50	40	32	25

Seb is investigating whether the relationship between temperature and time can be modelled by an equation of the form $\theta = Ab^t$.

a By taking logarithms to base e of both sides, show that the model can be written as $\ln\theta = \ln A + t\ln b$.

b Explain why, if the model is valid, plotting the graph of $\ln\theta$ against t will result in a straight line.

c Plot the graph of $\ln\theta$ against t and use it to estimate the values of A and b. Hence express the relationship in the form $\theta = Ab^t$.

d Why will this relationship not continue indefinitely?

Solution

a $\theta = Ab^t \Rightarrow \ln\theta = \ln Ab^t$

$$\Rightarrow \ln\theta = \ln A + \ln b^t$$

$$\Rightarrow \ln\theta = \ln A + t\ln b$$

b Rewriting $\ln\theta = \ln A + t\ln b$ as $\ln\theta = (\ln b)t + \ln A$ and comparing it with the equation $y = mx + c$ shows that plotting $\ln\theta$ against t will give a straight line with gradient $\ln b$ and intercept on the vertical axis at $\ln A$.

c

t	0	4	8	12	16	20
θ	80	63	50	40	32	25
$\ln\theta$	4.38	4.14	3.91	3.69	3.47	3.22