7 Logarithmic and exponential functions

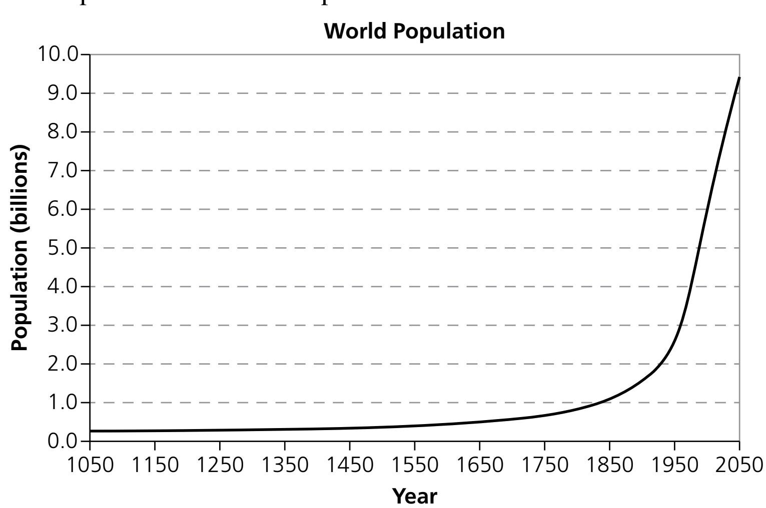
To forget one's ancestors is to be a brook without a source, a tree without a root.

Chinesepr overb

Discussion point

You have two parents and each of them has (or had) two parents so you have four grandparents. Going back you had $2^3 = 8$ great grandparents, $2^4 = 16$ great great grandparents and so on going backwards in time. Assuming that there is one generation every 30 years, and that all your ancestors were different people, estimate how many ancestors you had living in the year 1700. What about the year 1000?

The graph below shows an estimate of the world population over the last 1000 years. Explain why your answers are not realistic. What assumption has caused the problem?



Recent DNA analysis shows that almost everyone in Europe is descended from just seven women. Arriving at different times during the last 45 000 years, they survived wolves, bears and ice ages to form different clans that eventually became today's population. Another 26 maternal lineages have been uncovered on other continents.

Researching family history is a popular hobby and there are many Internet sites devoted to helping people find out details of their ancestry. Most of us only know about parents, grandparents, and possibly great-grandparents, but it is possible to go back much further.

Worked example

Assuming that a new generation occurs, on average, every 30 years, how many direct ancestors will be on your family tree if you go back 120 years? What about if you were able to go back 300 years?

Solution

30 years ago, you would have information about your two parents.

Each of these would have had two parents, so going back a further 30 years there are also four grandparents, another 30 years gives eight great-grandparents and so on.

If you tabulate these results, you can see a sequence starting to form.

Number of years	Number of people
30	2
60	$4 = 2^2$
90	$8 = 2^3$
120	$16 = 2^4$

This is a **geometric sequence** of numbers. You will meet these sequences in Chapter 12.

For each period of 30 years, the number of direct ancestors is double the number in the previous generation. After 120 years, the total number of ancestors is $2 + 4 + 8 + 16 = 30$.

300 years ago is ten periods of 30 years, so following the pattern, there are $2^{10} = 1024$ direct ancestors in this generation.

In practice, family trees are much more complicated, since most families have more than one child. It gets increasingly difficult the further back in time you research.

Discussion point

How many years would you expect to need to go back to find over 1 billion direct ancestors?

What date would that be?

Look at the graph on the previous page and say why this is not a reasonable answer.

Where has the argument gone wrong?

You may have answered the discussion point by continuing the pattern in the table at the top of the page. Or you may have looked for the smallest value of n- for which 2^n is greater than 1 billion. You can find this by trial and error but, as you will see, it is quicker to use logarithms to solve equations and inequalities like this.

Logarithms

Logarithm is another word for **index** or **power**.

For example, if you want to find the value of x such that $2^x = 8$, you can do this by checking powers of 2. However, if you have $2^x = 12$, for example, it is not as straightforward and you would probably need to resort to trial and improvement.

The equation $2^3 = 8$ can also be written as $\log_2 8 = 3$. The number 2 is referred to as the **base** of the logarithm.

Read this as 'log to base 2 of 8 equals 3'.

Similarly, $2^x = 12$ can be written as $\log_2 12 = x$.

In general,

$$a^x = y \Leftrightarrow x = \log_a y.$$

Most calculators have three buttons for logarithms.

- **log** which uses 10 as the base.
- **ln** which has as its base the number 2.718…, denoted by the letter e, which you will meet later in the chapter.
- **log**$_{\blacksquare}\square$ which allows you to choose your own base.

Worked example

Find the logarithm to base 2 of each of these numbers. Do not use a calculator.

a 32 **b** $\frac{1}{4}$ **c** 1 **d** $\sqrt{2}$

Solution

This is equivalent to being asked to find the power when the number is written as a power of 2.

a $32 = 2^5$, so $\log_2 32 = 5$

b $\frac{1}{4} = 2^{-2}$, so $\log_2 \frac{1}{4} = -2$

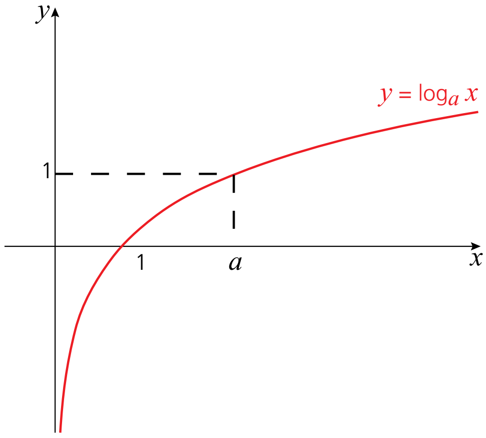
c $1 = 2^0$, so $\log_2 1 = 0$

$\log_n 1 = 0$ for all positive values of n.

d $\sqrt{2} = 2^{\frac{1}{2}}$, so $\log_2 \sqrt{2} = \frac{1}{2}$

Graphs of logarithms

The graph of $y = \log_a x$ has the same general shape for all values of the base a where $a > 1$.



The graph has the following properties:

- The curve only exists for positive values of x.
- The gradient of the graph is always positive. As the value of x increases, the gradient of the curve decreases.
- It crosses the x-axis at $(1, 0)$.
- The line $x = 0$ is an asymptote, i.e. the curve approaches it ever more closely but never actually touches or crosses it.
- The graph passes through the point $(a, 1)$.
- $\log_a x$ is negative for $0 < x < 1$.

Graphs of other logarithmic functions are obtained from this basic graph by applying one or more transformations – translations, stretches or reflections – as shown in the following examples.

Note

- A **translation** moves the graph – horizontally, vertically or in both directions – to a different position. It does not change in shape.
  When a > 0:
  - replacing x by $(x - a)$ moves the graph a units to the right (the positive direction)
  - replacing x by $(x + a)$ moves the graph a units to the left (the negative direction)
  - replacing y by $(y - a)$ moves the graph a units upwards (the positive direction)
  - replacing y by $(y + a)$ moves the graph a units downwards (the negative direction).
- A **reflection** gives a mirror image. In this book only reflections in the coordinate axes are considered.
  - Replacing x by $(- x)$ reflects the graph in the y-axis.
  - Replacing y by $(- y)$ reflects the graph in the x-axis.

Worked example

Sketch each pair of graphs and describe the transformation shown. In each pair, join (2, log 2) and (3, log 3) to their images.

a $y = \log x$ and $y = \log(x - 3)$

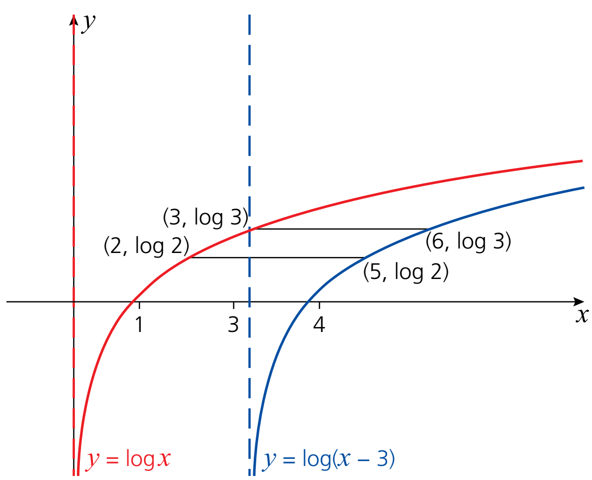
b $y = \log x$ and $y = \log(x + 2)$

c $y = \log x$ and $y = -\log x$

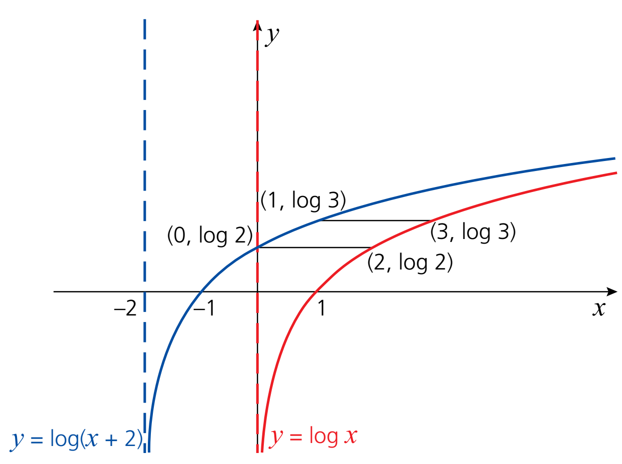
d $y = \log x$ and $y = \log(-x)$

Solution

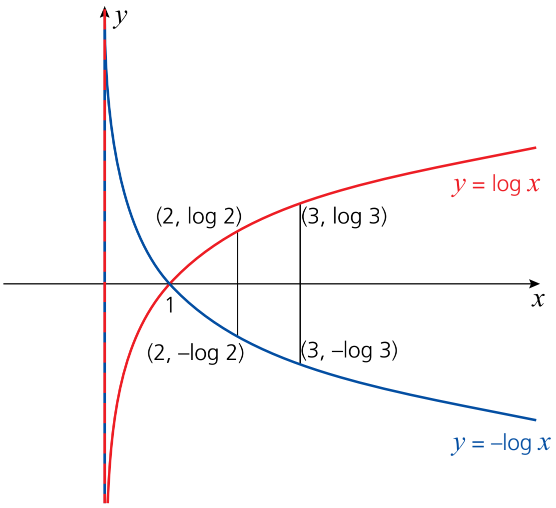
a The graph of $y = \log(x - 3)$ is a translation of the graph of $y = \log x$ 3 units to the right.



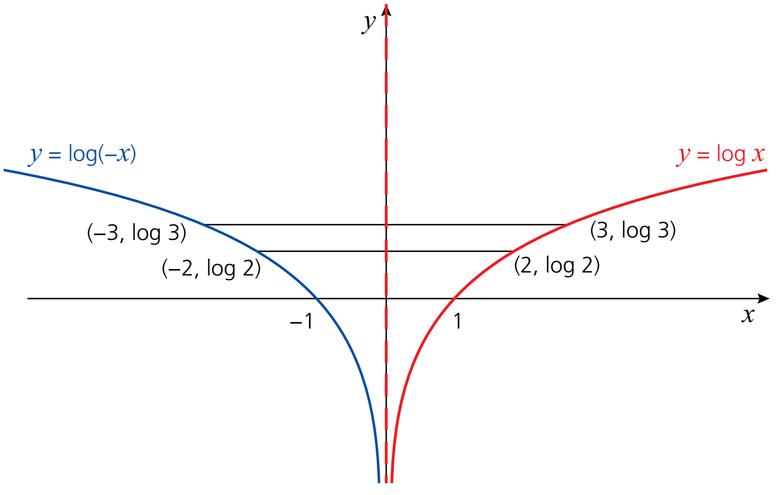
b The graph of $y = \log(x + 2)$ is a translation of the graph of $y = \log x$ 2 units to the left.



c The graph of $y = -\log x$ (which is the same as $-y = \log x$) is a reflection of the graph of $y = \log x$ in the x-axis.



d The graph of $y = \log(-x)$ is a reflection of the graph of $y = \log x$ in the y-axis.



Worked example

You are given the curve of $y = \log x$ and told that $\log 3 = 0.48$ (2 d.p.).

a Sketch the graph of $y = \log 3 + \log x$.

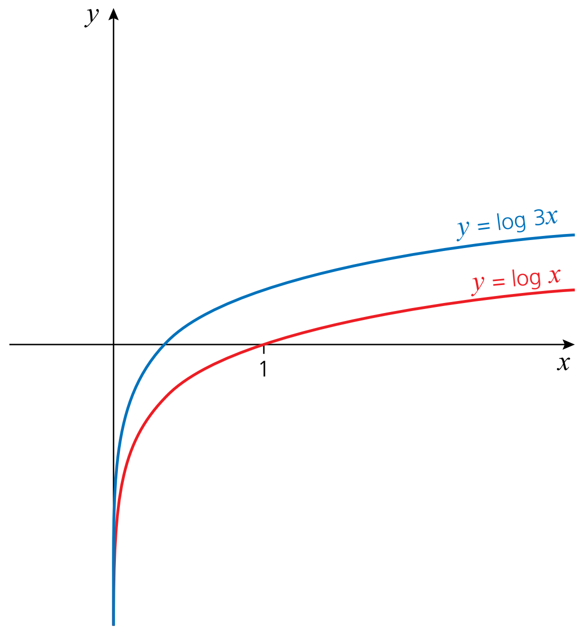
b What is the relationship between the graphs of $y = \log x$ and $y = \log 3 + \log x$?

c Sketch the graphs of $y = \log x$ and $y = \log 3x$ on the same axes.

d What do you notice?

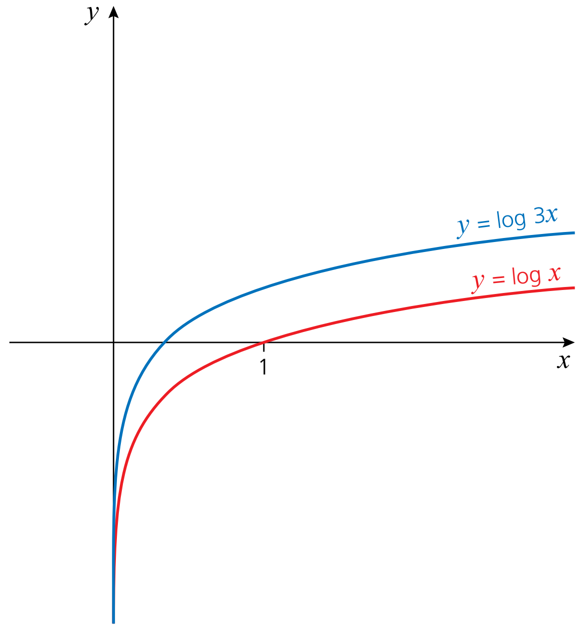
Solution

a



b The graph of $y = \log 3 + \log x$ is a translation of the graph of $y = \log x$

c upwards by a distance of log 3.



You can use graphing software to show that the graph of $y = \log 3x$ is the same as the graph of $y = \log 3 + \log x$. This confirms one of the 'laws of logarithms' introduced below.

d The graph of $y = \log 3x$ looks the same as the graph of $y = \log 3 + \log x$.

Laws of logarithms

If a logarithmic expression is true for any base, the base is often omitted.

There are a number of rules for manipulating logarithms. They are derived from the rules for manipulating indices. These laws are true for all logarithms to any positive base.

Operation	Law for indices	Law for logarithms
Multiplication	$a^x \times a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
Division	$a^x \div a^y = a^{x-y}$	$\log_a \frac{x}{y} = \log_a x - \log_a y$
Powers	$(a^x)^n = a^{nx}$	$\log_a x^n = n\log_a x$
Roots	$(a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$	$\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$
Logarithm of 1	$a^0 = 1$	$\log_a 1 = 0$
Reciprocals	$\frac{1}{a^x} = a^{-x}$	$\log_a \frac{1}{x} = \log_a 1 - \log_a x = -\log_a x$
Log to its own base	$a^1 = a$	$\log_a a = 1$

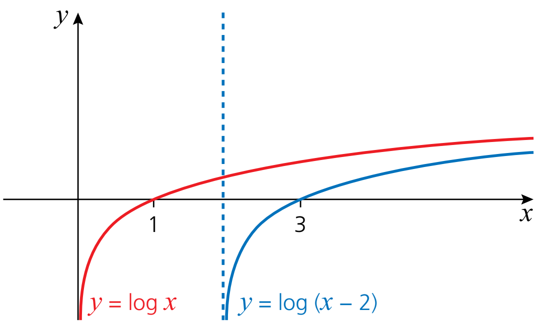
You can use these laws, together with the earlier work on translations, to help you sketch the graphs of a range of logarithmic expressions by breaking them down into small steps as shown below.

Worked example

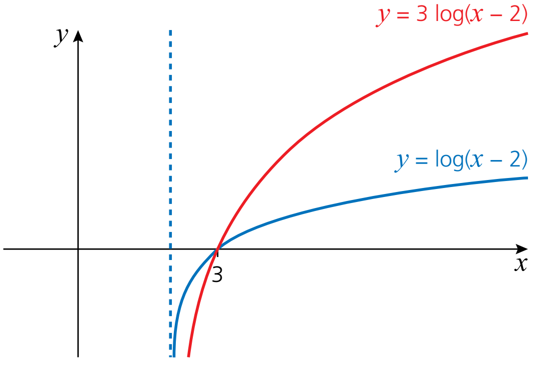
Sketch the graph of $y = 3\log(x-2)$.

Solution

Transforming the graph of the curve $y = \log x$ into $y = 3\log(x-2)$ involves two stages. Translating the graph of $y = \log x$ two units to the right gives the graph of $y = \log(x-2)$.



Multiplying log(x – 2) by 3 stretches the new graph in the y direction by a **scale factor** of 3.



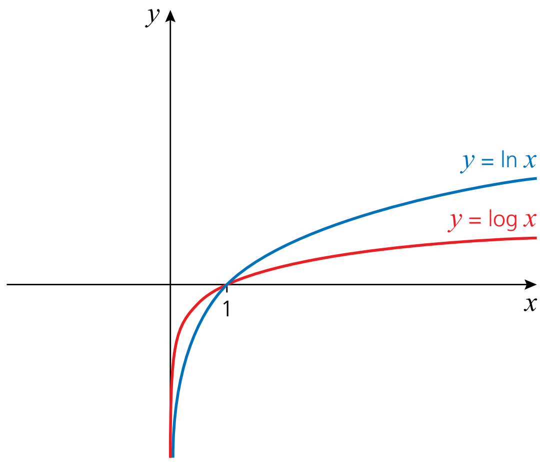
Logarithms to different bases

All the graphs you have met so far in the chapter could have been drawn to any base a greater than 1.

- When a logarithm is to the base 10 it can be written either as $\log_{10}$ or as lg. So, for example lg7 means $\log_{10}7$.
- Base e is the other common base for logarithms.

The graphs of logarithms with a base number that is not 10 are very similar to the graphs of logarithms with base 10.

Notice that when you use a different base for the logarithm, the graph has a similar shape and still passes through the point (1, 0).



Change of base of logarithms

It is sometimes useful to change the base of a logarithm.

$$x = \log_a b \Leftrightarrow a^x = b$$
$$\Leftrightarrow \log_c a^x = \log_c b$$
$$\Leftrightarrow x\log_c a = \log_c b$$
$$\Leftrightarrow x = \frac{\log_c b}{\log_c a}$$

Some calculators can manipulate logarithms to any positive base. Check whether yours is one of them.

Using logarithms to solve equations

Logarithms can be used to solve equations involving powers, to any level of accuracy.

This question does not ask for any particular base. In this case base 10 is used but you could alternatively have used base e. These are the two bases for logarithms on nearly all calculators.

Worked example

Solve the equation $3^x = 2000$.

Solution

Taking logarithms to the base 10 of both sides:

$$\lg 3^x = \lg 2000 \quad \Rightarrow \quad x\lg 3 = \lg 2000$$

$$\Rightarrow x = \frac{\lg 2000}{\lg 3} = 6.92 \text{ (3 s.f.)}$$

Logarithms can also be used to solve more complex equations.

Worked example

Solve the equation $4e^{3x} = 950$.

Solution

$$\begin{aligned} & 4e^{3x} = 950 \\ \Rightarrow\ & e^{3x} = 237.5 \\ \Rightarrow\ & 3x = \ln 237.5 \\ \Rightarrow\ & 3x = 5.470\,167\ldots \\ \Rightarrow\ & x = 1.82 \text{ (3 s.f.)} \end{aligned}$$

Taking the logarithms to base e of both sides

When there is a term of the form e^x, it is easier to use logarithms to base e, i.e. the ln button on your calculator.

Worked example

Use logarithms to solve the equation $3^{5-x} = 2^{5+x}$. Give your answer correct to 3 s.f.

Solution

No base is mentioned, so you can use logarithms to any base. Using base 10:

$$\begin{aligned} & 3^{5-x} = 2^{5+x} \\ \Rightarrow\ & \lg 3^{5-x} = \lg 2^{5+x} \\ \Rightarrow\ & (5-x)\lg 3 = (5+x)\lg 2 \\ \Rightarrow\ & 5\lg 3 - x\lg 3 = 5\lg 2 + x\lg 2 \\ \Rightarrow\ & 5\lg 3 - 5\lg 2 = x\lg 2 + x\lg 3 \\ \Rightarrow\ & 5(\lg 3 - \lg 2) = x(\lg 2 + \lg 3) \\ \Rightarrow\ & x = \frac{5(\lg 3 - \lg 2)}{(\lg 2 + \lg 3)} \\ \Rightarrow\ & x = 1.13 \end{aligned}$$

Note that any base will yield the same answer. Using base 2:

$$\begin{aligned} & 3^{5-x} = 2^{5+x} \\ \Rightarrow\ & \log_2 3^{5-x} = \log_2 2^{5+x} \\ \Rightarrow\ & (5-x)\log_2 3 = (5+x)\log_2 2 \\ \Rightarrow\ & 5\log_2 3 - x\log_2 3 = 5 + x \\ \Rightarrow\ & 5\log_2 3 - 5 = x + x\log_2 3 \\ \Rightarrow\ & 5(\log_2 3 - 1) = x(1 + \log_2 3) \\ \Rightarrow\ & x = \frac{5(\log_2 3 - 1)}{1 + \log_2 3} \\ \Rightarrow\ & x = 1.13 \text{ (3 s.f.)} \end{aligned}$$

Remember, $\log_2 2 = 1$.

Discussion point

Did you find one of these methods easier than the other?
If so, which one?

Using logarithms to solve inequalities

Logarithms are also useful to solve inequalities occurring, for example, in problems involving interest or depreciation.

When an inequality involves logs, it is often better to solve it as an equation first and then address the inequality. If you choose to solve it as an inequality, you may need to divide by a negative quantity and will therefore need to reverse the direction of the inequality sign. Both methods are shown in the example below.

Worked example

A second-hand car is bought for \$20 000 and is expected to depreciate at a rate of 15% each year. After how many years will it first be worth less than \$10 000?

Solution

The rate of depreciation is 15% so after one year the car will be worth 85% of the initial cost.

At the end of the second year, it will be worth 85% of its value at the end of Year 1, so $(0.85)^2 \times \$20000$.

Continuing in this way, its value after n years will be $(0.85)^n \times \$20000$.

Method 1: Solving as an equation

Solving the equation $(0.85)^n \times 20000 = 5000$

$$\Rightarrow \quad (0.85)^n = 0.5$$

$$\Rightarrow \quad \lg 0.85^n = \lg 0.5$$

$$\Rightarrow \quad n \lg 0.85 = \lg 0.5$$

$$\Rightarrow \quad n = \frac{\lg 0.5}{\lg 0.85}$$

$$\Rightarrow \quad n = 4.265\ldots$$

The car will be worth \$10 000 after 4.265 years, so it is **5 years** before it is first worth less than \$10 000.

Remember that lg 0.85 is negative and when you divide an inequality by a negative number, you must change the direction of the inequality.

Method 2: Solving as an inequality

Solving the inequality $(0.85)^n \times 20\,000 < 10\,000$

$$\Rightarrow (0.85)^n < 0.5$$

$$\Rightarrow \lg 0.85^n < \lg 0.5$$

$$\Rightarrow n \lg 0.85 < \lg 0.5$$

$$\Rightarrow n > \frac{\lg 0.5}{\lg 0.85}$$

$$\Rightarrow n > 4.265\ldots$$

The car will be worth less than \$10 000 after 5 years.

Exercise 7.1

In some of the following questions you are instructed not to use your calculator for the working, but you may use it to check your answers.

1 By first writing each of the following equations using powers, find the value of y without using a calculator:

a $y = \log_2 8$ **b** $y = \log_3 1$ **c** $y = \log_5 25$ **d** $y = \log_2 \frac{1}{4}$

2 $3^2 = 9$ can be written using logarithms as $\log_3 9 = 2$. Using your knowledge of indices, find the value of each of the following without using a calculator:

a $\log_2 16$ **b** $\log_3 81$ **c** $\log_5 125$ **d** $\log_4 \frac{1}{64}$

Remember that lg means $\log_{10}$.

3 Find the following without using a calculator:

a $\lg 100$ **b** lg(one million)

c $\lg \frac{1}{1000}$ **d** $\lg(0.000\,001)$

4 Using the rules for manipulating logarithms, rewrite each of the following as a single logarithm. For example, $\log 6 + \log 2 = \log(6 \times 2) = \log 12$.

a $\log 3 + \log 5$ **b** $3 \log 4$

c $\log 12 - \log 3$ **d** $\frac{1}{2} \log 25$

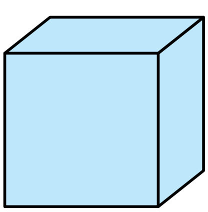
e $2 \log 3 + 3 \log 2$ **f** $4 \log 3 - 3 \log 4$

g $\frac{1}{2} \log 4 + 4 \log \frac{1}{2}$

5 Express each of the following in terms of $\log x$:

a $\log x^5 - \log x^2$ **b** $\log x^3 + 3 \log x$ **c** $5 \log \sqrt{x} - 3 \log \sqrt[3]{x}$

6 This cube has a volume of $800\,\text{cm}^3$.



a Use logarithms to calculate the side length correct to the nearest millimetre.

b What is the surface area of the cube?

Exercise 7.1 (cont)

7 Starting with the graph of $y = \ln x$, list the transformations required, in order when more than one is needed, to sketch each of the graphs. Use the transformations you have listed to sketch each graph.

a $y = 3\ln x$ **b** $y = \ln(x + 3)$ **c** $y = 3\ln 2x$

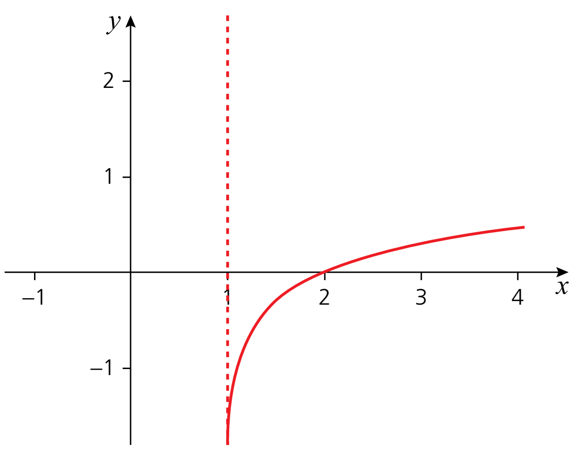
d $y = 3\ln x + 2$ **e** $y = -3\ln(x + 1)$ **f** $y = \ln(2x + 4)$

8 Match each equation from **i** to **vi** with the correct graph **a** to **f**.

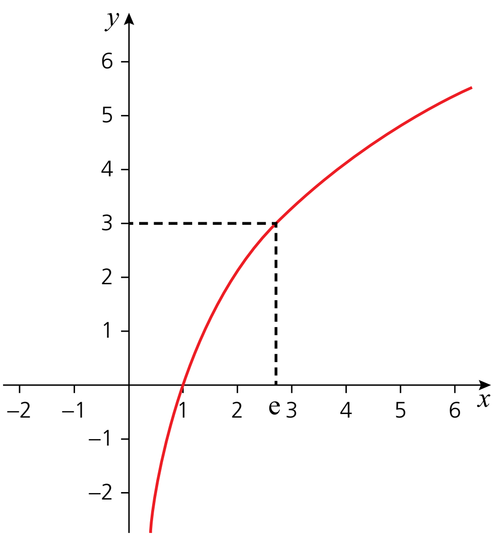
i $y = \log(x + 1)$ **ii** $y = \log(x - 1)$ **iii** $y = -\ln x$

iv $y = 3\ln x$ **v** $y = \log(2 - x)$ **vi** $y = \ln(x + 2)$

a



b



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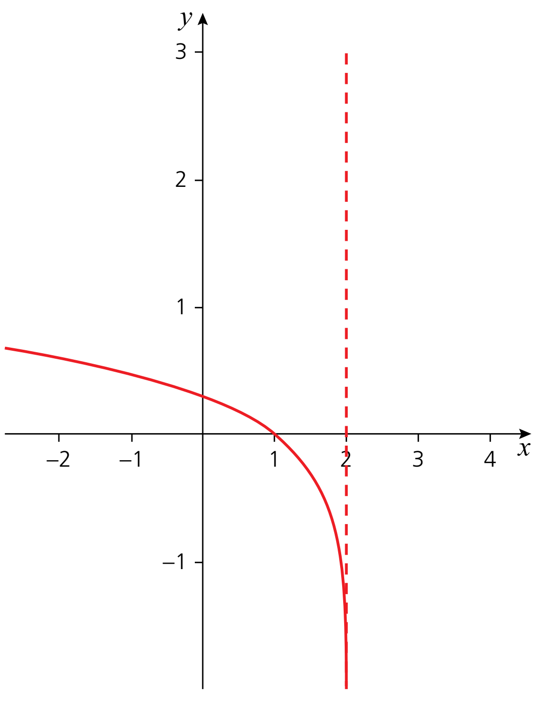

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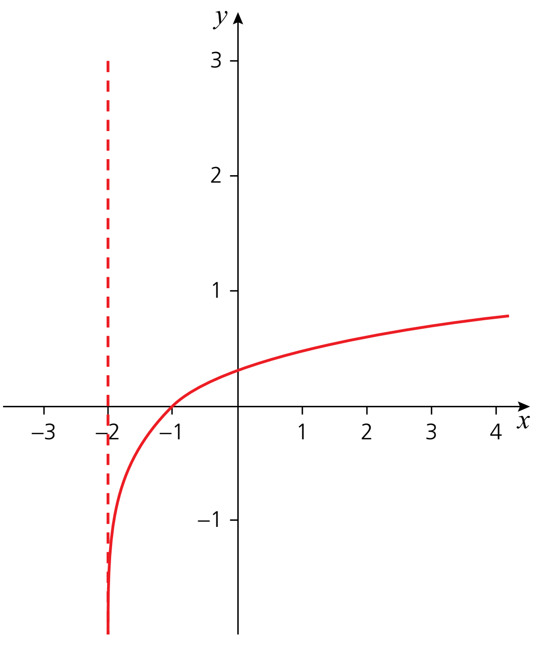
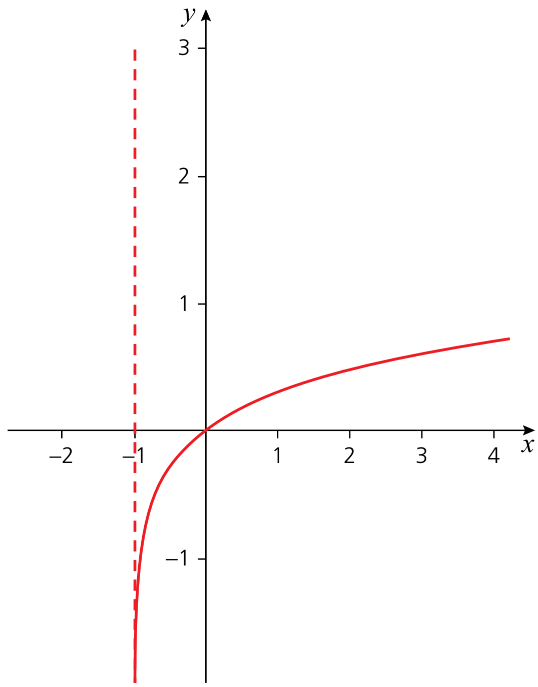
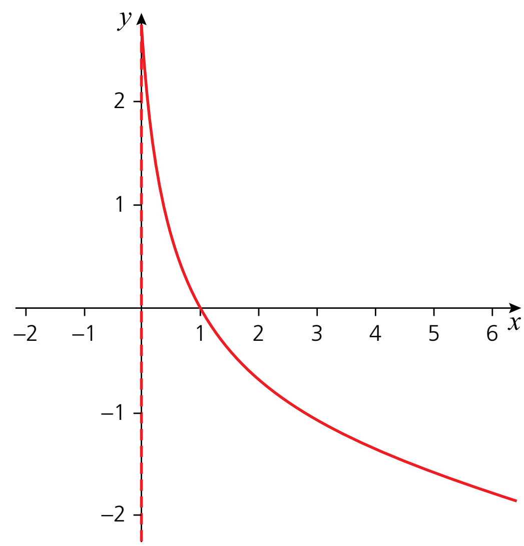


e



f



9 Solve the following equations for x, given that $\ln a = 3$:

a $a^{2x} = e^3$

b $a^{3x} = e^2$

c $a^{2x} - 3a^x + 2 = 0$

10 Before photocopiers were commonplace, school examination papers were duplicated using a process where each copy produced was only c% as clear as the previous copy. The copy was not acceptable if the writing was less than 50% as clear as the original. What is the value of c if the machine could produce only 100 acceptable copies from the original?

11 Use logarithms to solve the equation $5^{2x-1} = 4^{x+3}$. Give the value of x correct to 3 s.f.

12 a \$20 000 is invested in an account that pays interest at 2.4% per annum. The interest is added at the end of each year. After how many years will the value of the account first be greater than \$25 000?

b What percentage interest should be added each month if interest is to be accrued monthly?

c How long would the account take to reach \$25 000 if the interest was added:

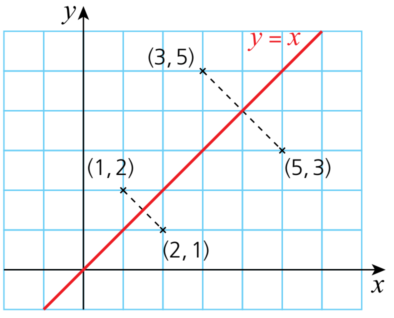
i every month

ii every day?

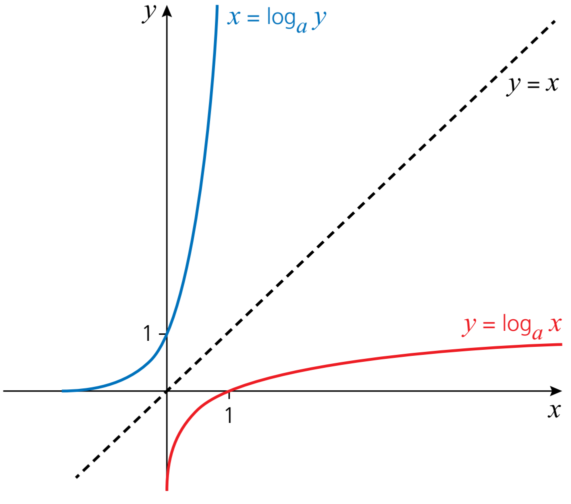
Exponential functions

The expression $y = \log_a x$ can be written as $x = a^y$. Therefore, the graphs of these two expressions are identical.

For any point, interchanging the x- and y-coordinates has the effect of reflecting the original point in the line $y = x$, as shown below.



Interchanging x and y for the graph $y = \log_a x$ (shown in red) gives the curve $x = \log_a y$ (shown in blue).



When rewritten with y as the subject of the equation, $x = \log_a y$ becomes $y = a^x$.

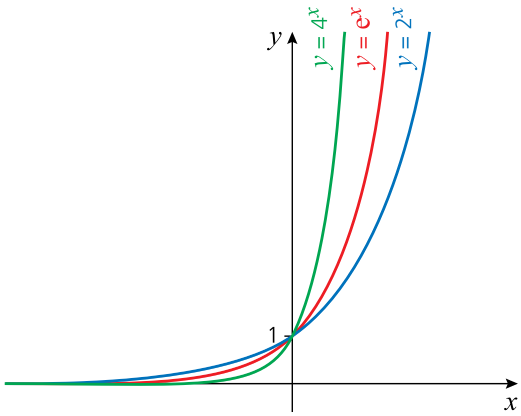
The function $y = a^x$ is called an **exponential function** and is the **inverse** of the logarithm function.

The most commonly used exponential function, known as **the exponential function**, is e^x, where e is the base of the logarithmic function $\ln x$ and is approximately equal to 2.718. You can manipulate exponential functions using the same rules as any other functions involving powers.

- $e^{a+b} = e^a \times e^b$
- $e^{a-b} = e^a \div e^b$

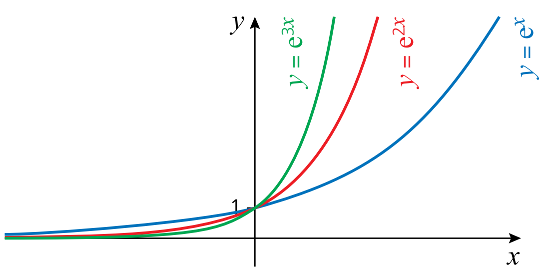
Graphs of e^x and associated exponential functions

The graph of $y = e^x$ has a similar shape to the graph of $y = a^x$ for positive value of a. The difference lies in the steepness of the curve.

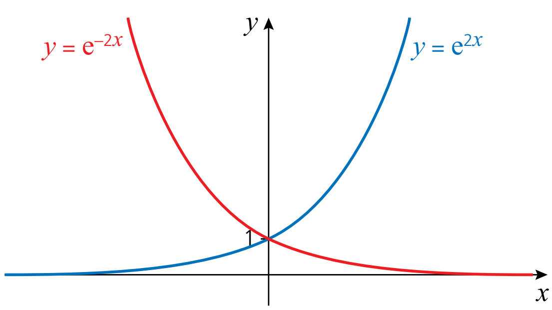


As the base number increases (i.e. 2, e and 4 in the equations above), the curve becomes steeper for positive values of x. All the y-values are positive and all the curves pass through and 'cross over' at the point (0, 1).

For positive integer values of n, curves of the form $y = e^{nx}$ are all related as shown below. Notice again, that the graphs all pass through the point (0, 1) and, as the value of n increases, the curves become steeper.

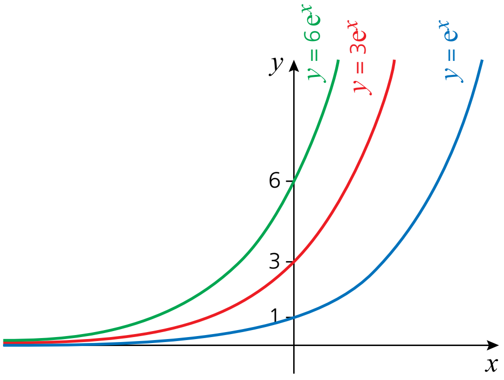


The graph of $y = e^{-x}$ is a reflection in the y-axis of the graph of $y = e^x$. The graphs of $y = e^{nx}$ and $y = e^{-nx}$ are related in a similar way for any integer value of n.



The family of curves $y = ke^x$, where k is a positive integer, is different set of transformations of the curve $y = e^x$. These represent stretches of the curve $y = e^x$ in the y-direction.

Notice that the curve $y = ke^x$ crosses the y-axis at $(0, k)$.



Similarly, for a fixed value of n, graphs of the family $y = ke^{nx}$ are represented by stretches of the graph $y = e^{nx}$ by scale factor k in the y-direction.

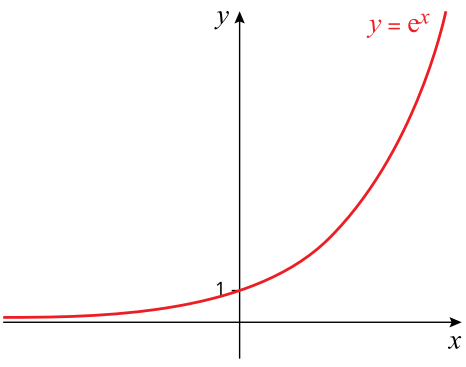
One additional transformation gives graphs of the form $y = ke^{nx} + a$.

Worked example

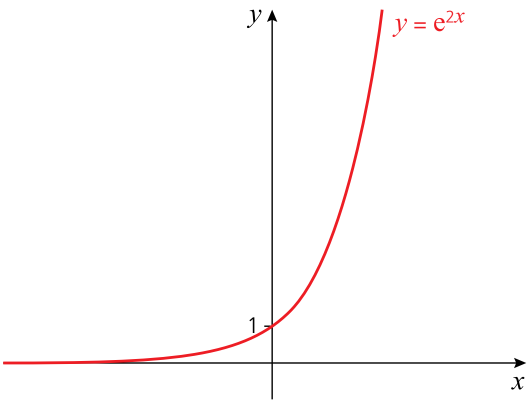
Sketch the graph of $y = 3e^{2x} + 1$.

Solution

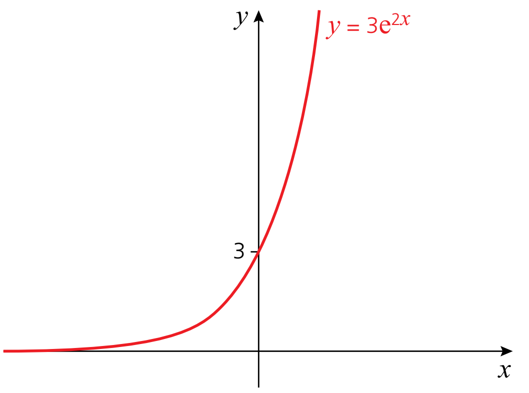
Start with $y = e^x$.



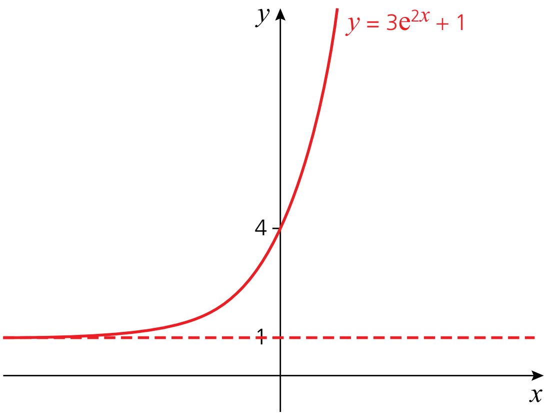
Transform to $y = e^{2x} = (e^x)^2$.
The y values are squared, giving smaller values for $x < 0$ (where $y < 1$) and larger values for $x > 0$.



Stretch in the y-direction with a scale factor of 3 to give the graph of $y = 3e^{2x}$.



Translate 1 unit upwards to give $y = 3e^{2x} + 1$.



Worked example

a Solve the equation $e^{2x} - 5e^{x} + 6 = 0$

b Hence solve the equation $e^{4x} - 5e^{2x} + 6 = 0$

Solution

a $e^{2x} - 5e^{x} + 6 = 0$

$\Rightarrow (e^{x} - 2)(e^{x} - 3) = 0$

$\Rightarrow e^{x} = 2 \text{ or } e^{x} = 3$

$\Rightarrow x = \ln 2 \text{ or } x = \ln 3$

$\Rightarrow x = 0.693 \text{ or } x = 1.099 \text{ (3 d.p.)}$

b $e^{4x} - 5e^{2x} + 6 = (e^{2x} - 2)(e^{2x} - 3)$

So, either $2x = \ln 2 \Rightarrow x = 0.347$ (3 d.p.)

or $2x = \ln 3 \Rightarrow x = 0.549$ (3 d.p.)

Exponential growth and decay

The word 'exponential' is often used to refer to things that increase or decrease at a very rapid rate.

$y = e^x$ is called **the** exponential function.

Any function of the form $y = a^x$ is referred to as an exponential function. When $x > 0$, the function $y = a^x$ is referred to as **exponential growth**; when $x < 0$ it is **exponential decay**.

Worked example

During the growth of an organism, a cell divides into two approximately every 6 hours. Assuming that the process starts with a single cell, and none of the cells die, how many cells will there be after 1 week?

Solution

It is possible to work this out without any special formulae:

2 cells after 6 hours

4 cells after 12 hours

8 cells after 18 hours…

However as the numbers get larger, the working becomes more tedious.

Notice the pattern here using 6 hours as 1 time unit.

2^1 cells after 1 time unit

2^2 cells after 2 time units

2^3 cells after 3 time units…

1 day of 24 hours is 4 time units, so 1 week of 7 days is 28 time units. So after 1 week there will be $2^{28} = 268\,435\,456$ cells.

Worked example

A brand of 'invisible' ink fades rapidly once it is applied to paper. After each minute the intensity is reduced by one quarter. It becomes unreadable to the naked eye when the intensity falls below 5% of the original value.

a What is the intensity, as a percentage of the original value, after 3 minutes?

b After how many minutes does it become unreadable to the naked eye? Give your answer to the nearest whole number.

Solution

a After 1 minute it is $\frac{3}{4}$ of the original value.

After 2 minutes it is $\frac{3}{4}\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^2$ of the original value.

After 3 minutes it is $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$ or approximately 42% of the original value.

It would be very tedious to continue the method in used above until the ink becomes unreadable.

b Using the pattern developed above:

After t minutes it is approximately $\left(\frac{3}{4}\right)^t$ of the original value.

The situation is represented by: $\left(\frac{3}{4}\right)^t < \frac{5}{100}$ ← $5\% = \frac{5}{100}$

Using logarithms to solve the inequality as an equation:

$$\lg\left(\frac{3}{4}\right)^t = \lg\frac{5}{100} \Rightarrow t\lg\left(\frac{3}{4}\right) = \lg\left(\frac{5}{100}\right)$$
$$\Rightarrow t\lg 0.75 = \lg 0.05$$
$$\Rightarrow t = \frac{\lg 0.05}{\lg 0.75}$$
$$\Rightarrow t = 10.4$$

Since the question asks for the time as a whole number of minutes, and the time is increasing, the answer is 11 minutes.

Exercise 7.2

It is a good idea to check the graphs you draw in Questions 1–4 using any available graphing software.

1 For each set of graphs:
  i Sketch the graphs on the same axes.
  ii Give the coordinates of any points of intersection with the axes.
    a $y = e^x$, $y = e^x + 1$ and $y = e^{x+1}$
    b $y = e^x$, $y = 2e^x$ and $y = e^{2x}$
    c $y = e^x$, $y = e^x - 3$ and $y = e^{x-3}$

2 Sketch the graphs of $y = e^{3x}$ and $y = e^{3x} - 2$.

3 Sketch the graphs of $y = e^{2x}$, $y = 3e^{2x}$ and $y = 3e^{2x} - 1$.

4 Sketch each curve and give the coordinates of any points where it cuts the y-axis.
  a $y = 2 + e^x$ **b** $y = 2 - e^x$ **c** $y = 2 + e^{-x}$ **d** $y = 2 - e^{-x}$

5 Solve the following equations:
  a $5e^{0.3t} = 65$ **b** $13e^{0.5t} = 65$ **c** $e^{t+2} = 10$ **d** $e^{t-2} = 10$

6 The value, \$$V$, of an investment after t years is given by the formula $V = Ae^{0.03t}$, where \$$A$ is the initial investment.
  a How much, to the nearest dollar, will an investment of \$4000 be worth after 3 years?
  b To the nearest year, how long will I need to keep an investment for it to double in value?

7 The path of a projectile launched from an aircraft is given by the equation $h = 5000 - e^{0.2t}$, where h is the height in metres and t is the time in seconds.
  a From what height was the projectile launched?

  The projectile is aimed at a target at ground level.
  b How long does it take to reach the target?

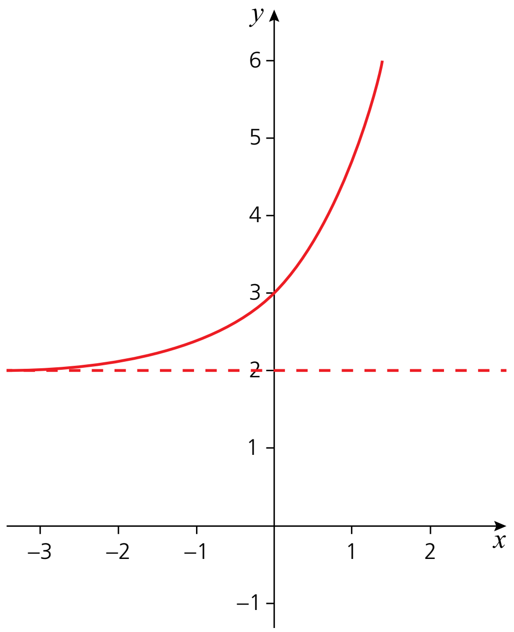
Exercise 7.2 (cont)

8 Match each equation from **i** to **vi** to the correct graph **a** to **f**.

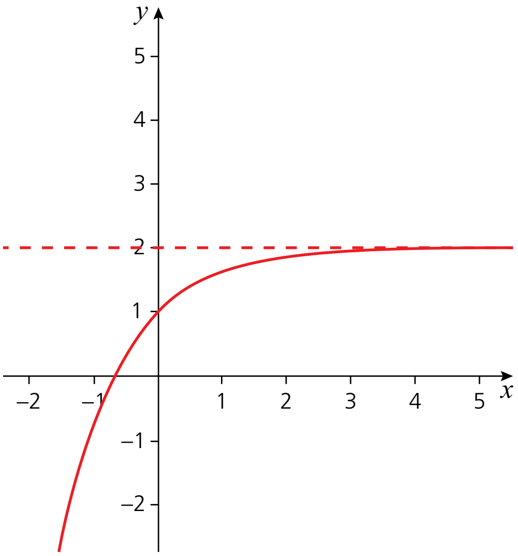
i $y = e^{2x}$ **ii** $y = e^{x} + 2$ **iii** $y = 2 - e^{x}$

iv $y = 2 - e^{-x}$ **v** $y = 3e^{-x} - 5$ **vi** $y = e^{-2x} - 1$

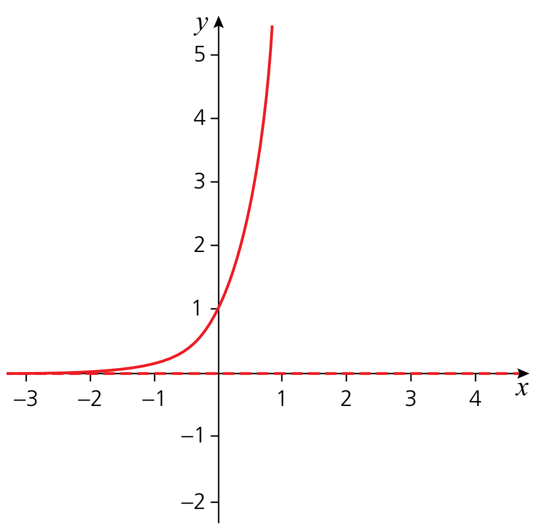
a



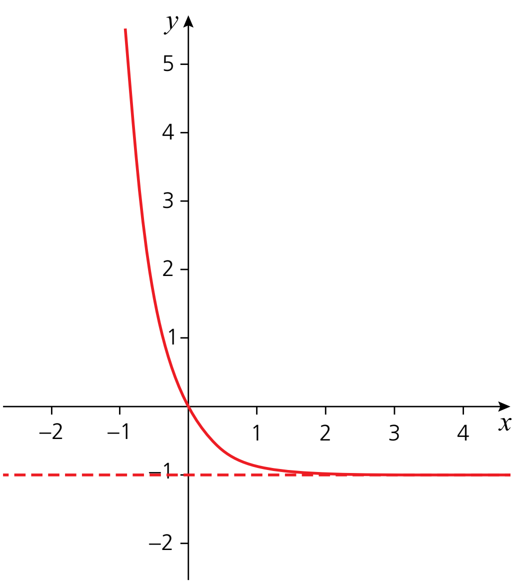
b



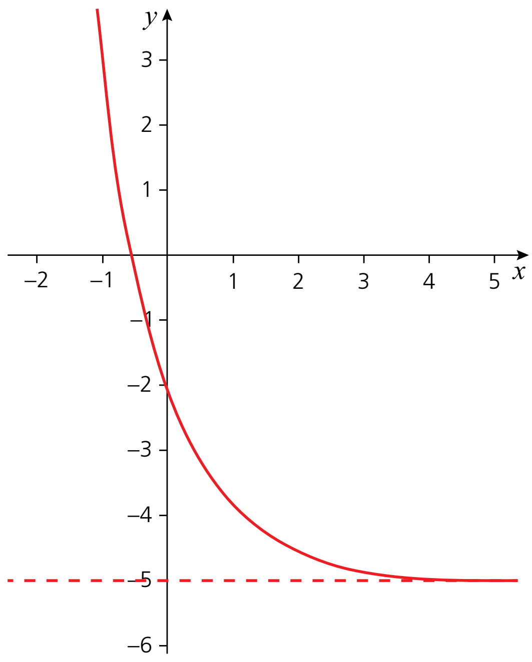
c



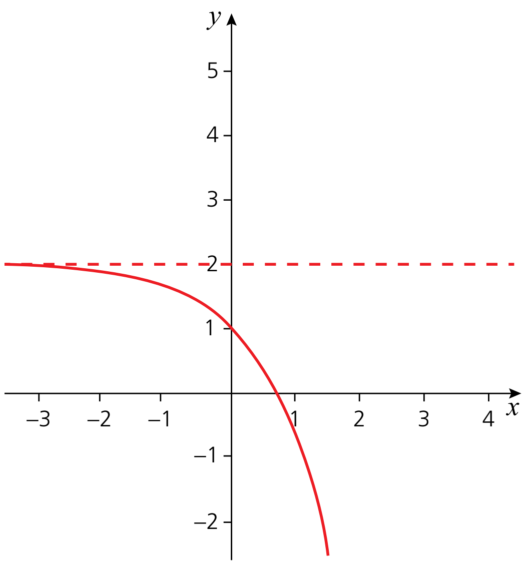
d



e



f



9 A radioactive substance of mass 100 g is decaying such that after t days the amount remaining, M, is given by the equation $M = 100e^{-0.002t}$.

a Sketch the graph of M against t.

b What is the half-life of the substance (i.e. the time taken to decay to half the initial mass)?

10 When David started his first job, he earned \$15 per hour and was promised an annual increment (compounded) of 3.5%.

a What is his hourly rate be in his 5th year?

After 5 years he was promoted. His hourly wage increased to \$26 per hour, with the same compounded annual increment.

b For how many more years will he need to work before his hourly rate reaches \$30 per hour?

11 a Solve $2(3^{2x}) - 5(3^x) + 2 = 0$

b Solve $e^x e^{x+1} = 10$

c Solve $2^{2x} - 5(2^x) + 4 = 0$

Past–paper questions

1 Given that $\log_a pq = 9$ and $\log_a p^2q = 15$, find the value of

(i) $\log_a p$ and of $\log_a q$, [4]

(ii) $\log_p a + \log_q a$. [2]

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Paper 12 Q4 November 2012
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2 Solve the simultaneous equations

$\log_3 a = 2\log_3 b$,

$\log_3 (2a - b) = 1$. [5]

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Paper 13 Q5 November 2010
Cambridge IGCSE Additional Mathematics 0606
Paper 13 Q5 November 2010

3 The number of bacteria B in a culture, t days after the first observation, is given by

$B = 500 + 400e^{0.2t}$

(i) Find the initial number present. [1]

(ii) Find the number present after 10 days. [1]

(iv) Find the value of t when $B = 10\,000$. [3]

Cambridge O Level Additional Mathematics 4037
Paper 22 Q5 November 2014
(Part question: part (iii) omitted)
Cambridge IGCSE Additional Mathematics 0606
Paper 22 Q5 November 2014
(Part question: part (iii) omitted)

Learning outcomes

Now you should be able to:

★ recognise simple properties and graphs of the logarithmic and exponential functions including $\ln x$ and e^x and graphs of $ke^{nx} + a$ and $k\ln(ax + b)$ where n, k, a and b are integers

★ recognise and use the laws of logarithms (including change of base of logarithms)

★ solve equations of the form $a^x = b$.

Key points

- **Logarithm** is another word for **index** or **power**.
- The laws for logarithms are valid for all bases greater than 0 and are related to those for indices.

Operation	Law for indices	Law for logarithms
Multiplication	$a^x \times a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
Division	$a^x \div a^y = a^{x-y}$	$\log_a \frac{x}{y} = \log_a x - \log_a y$
Powers	$(a^x)^n = a^{nx}$	$\log_a x^n = n\log_a x$
Roots	$(a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$	$\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$
Logarithm of 1	$a^0 = 1$	$\log_a 1 = 0$
Reciprocals	$\frac{1}{a^x} = a^{-x}$	$\log_a \frac{1}{x} = \log_a 1 - \log_a x = -\log_a x$
Log to its own base	$a^1 = a$	$\log_a a = 1$

- The graph of $y = \log x$: is only defined for $x > 0$
  has the y-axis as an asymptote
  has a positive gradient
  passes through $(0, 1)$ for all bases.
- Notation.
  The logarithm of x to the base a is written $\log_a x$.
  The logarithm of x to the base 10 is written $\lg x$ or $\log x$.
  The logarithm of x to the base e is written $\ln x$.
- An **exponential function** is of the form $y = a^x$.
- The exponential function is the inverse of the log function.
  $y = \log_a x \Leftrightarrow a^y = x$
- For $a > 0$, the graph of $y = a^x$: has the x-axis as an asymptote has a positive gradient passes through $(0, 1)$.
- For $a > 0$, the graph of $y = a^{-x}$: has the x-axis as an asymptote has a negative gradient passes through $(0, 1)$.