5 Factors of polynomials

A **quadratic expression** is any expression of the form $ax^2 + bx + c$, where x is a variable and a, b and c are constants with $a \neq 0$.
An expression of the form $ax^3 + bx^2 + cx + d$ that includes a term in x^3 is called a **cubic expression**.
A **quartic** has a term in x^4 as its highest power, a **quintic** one with x^5 and so on.
All of these are **polynomials** and the highest power of the variable is called the **order** of the polynomial.

There are things of an unknown number which when divided by 3 leave 2, by 5 leave 3, and by 7 leave 2. What is the smallest number?

Sun-Tzi (544 BC - 496 BC)

Discussion point

Sun Tzi posed his problem in the Chinese Han dynasty and it is seen as the forerunner of the Remainder Theorem, which you will meet in this chapter. What is the answer? What is the next possible answer to Sun-Tzi's problem? How do you find further answers?

It is believed that the way that numbers were written during the Han dynasty laid the foundation for the abacus, an early form of hand calculator.

In Chapter 2 you met quadratic expressions like $x^2 - 4x - 12$ and solved quadratic equations such as $x^2 - 4x - 12 = 0$.

Multiplication and division of polynomials

There are a number of methods for multiplying and dividing polynomials. One method is shown in the worked example, but if you already know and prefer an alternative method, continue to use it.

Multiplication

Worked example

Multiply (x^2-5x+2) by $(2x^2-x+1)$.

Solution

This is an extension of the method you used to multiply two brackets that each contain two terms. If you are familiar with a different method, then use that.

$$\begin{aligned}(x^2-5x+2)\times(2x^2-x+1) &= x^2(2x^2-x+1)-5x(2x^2-x+1)+2(2x^2-x+1)\\ &= 2x^4-x^3+x^2-10x^3+5x^2-5x+4x^2-2x+2\\ &= 2x^4+x^3(-1-10)+x^2(1+5+4)+x(-5-2)+2\\ &= 2x^4-11x^3+10x^2-7x+2\end{aligned}$$

Division

Worked example

Divide (x^3-x^2-2x+8) by $(x+2)$.

Solution

Let $(x^3-x^2-2x+8)=(x+2)(ax^2+bx+c)$

Multiplying each term in the second bracket by x and then by 2

$$=x(ax^2+bx+c)+2(ax^2+bx+c)$$

This bracket must be a quadratic expression.

$$=ax^3+bx^2+cx+2ax^2+2bx+2c$$

Collecting like terms

$$=ax^3+(b+2a)x^2+(c+2b)x+2c$$

Comparing coefficients:

$a=1$

Since $a=1$

$b+2a=-1\Rightarrow b=-3$

Since $b=-3$

$c+2b=-2\Rightarrow c=4$

Checking the constant term, $2c=8$ which is correct.

This gives $(x^3-x^2-2x+8)\div(x+2)=x^2-3x+4$

Exercise 5.1

1 Multiply $(x^3 + 2x^2 - 3x - 4)$ by $(x + 1)$.

2 Multiply $(x^3 - 2x^2 + 3x + 2)$ by $(x - 1)$.

3 Multiply $(2x^3 - 3x^2 + 5)$ by $(2x - 1)$.

4 Multiply $(x^2 + 2x - 3)$ by $(x^2 - 2x + 3)$.

5 Multiply $(2x^2 - 3x + 4)$ by $(2x^2 - 3x - 4)$.

6 Simplify $(x^2 - 3x + 2)^2$.

7 Divide $(x^3 - 3x^2 + x + 1)$ by $(x - 1)$.

8 Divide $(x^3 - 3x^2 + x + 2)$ by $(x - 2)$.

9 Divide $(x^4 - 1)$ by $(x + 1)$.

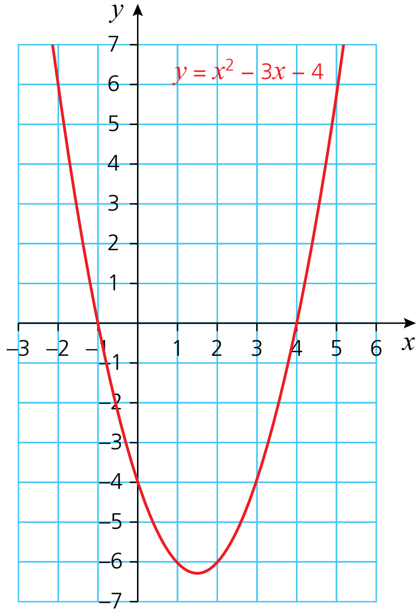
10 Divide $(x^2 - 16)$ by $(x + 2)$.

Solving cubic equations

When a polynomial can be factorised, you can find the points where the corresponding curve crosses the x-axis either as whole numbers or simple fractions.

For example, $y = x^2 - 3x - 4$ factorises to give $y = (x + 1)(x - 4)$.

The graph of this equation is a curve that crosses the x-axis at the points where $y = 0$. These values, $x = -1$ and $x = 4$, are called the **roots** of the equation $x^2 - 3x - 4 = 0$.



For a polynomial of the form $y = f(x)$, the roots are the solutions of $f(x) = 0$.

Worked example

a Draw the graph of $y = x^3 - 5x^2 + 2x + 8$.

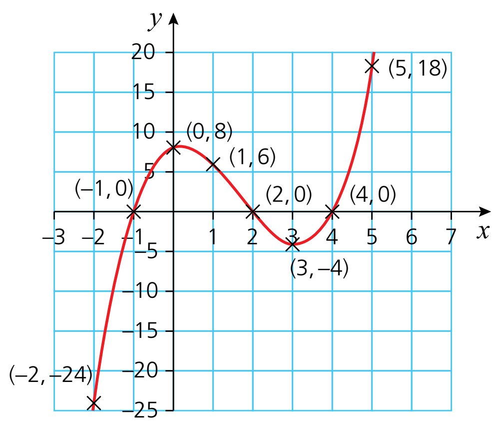
b Hence solve the equation $x^3 - 5x^2 + 2x + 8 = 0$.

Solution

a Start by setting up a table of values.

x	–2	–1	0	1	2	3	4	5
y	–24	0	8	6	0	–4	0	18

Then plot the curve.



The solution is '$x = -1$ or $x = 2$ or $x = 4$' but the roots are '–1 and 2 and 4'.

b The graph shows that the curve crosses the x-axis at the values –1, 2 and 4, giving the solution as $x = -1$, $x = 2$ or $x = 4$.

In some cases, a graph will not find all the roots but will allow you to find one or possibly two roots, or show you that there is only one root. The roots may not be whole numbers and may not even be rational as shown in the following examples.

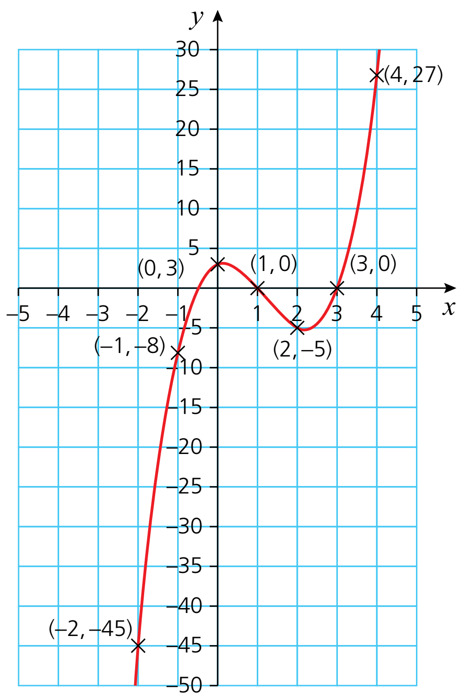
Worked example

Draw the graph of $y = 2x^3 - 7x^2 + 2x + 3$ and hence solve the equation $2x^3 - 7x^2 + 2x + 3 = 0$.

Solution

As before, start by setting up a table of values and then draw the curve.

x	–2	–1	0	1	2	3	4
y	–45	–8	3	0	–5	0	27



The graph shows that the curve crosses the x-axis at 1, at 3 and again between –1 and 0. You can find this root using trial and improvement.

–0.5 is chosen as x since it is half way between –1 and 0.

Let $x = -0.5$

$$f(0.5) = 2(-0.5)^3 - 7(-0.5)^2 + 2(-0.5) + 3$$
$$= 0$$

In this case, you were lucky and found the final root, –0.5, with only one iteration.

So the roots of the equation are –0.5, 1 and 3.

Finding factors and the factor theorem

The equation in the example above has roots that are whole numbers or exact fractions. This implies that it could have been factorised. Roots at $-\frac{1}{2}$, 1 and 3 suggest the factorised form:

$$\left(x+\tfrac{1}{2}\right)(x-1)(x-3)$$

However multiplying the x terms from all the brackets should give $2x^3$ so one of the brackets must be multiplied by 2.

$$2x^3 - 7x^2 + 2x + 3 = (2x+1)(x-1)(x-3)$$

It is not possible to factorise all polynomials. However, when a polynomial can be factorised, the solution to the corresponding equation follows immediately.

$$(2x+1)(x-1)(x-3) = 0 \quad \Rightarrow \quad (2x+1) = 0 \text{ or } (x-1) = 0 \text{ or } (x-3) = 0$$
$$\Rightarrow \quad x = -0.5 \text{ or } x = 1 \text{ or } x = 3$$

This leads to an important result known as the **factor theorem**.

> If $(x-a)$ is a factor of $f(x)$, then $f(a)=0$ and $x=a$ is a root of the equation $f(x)=0$.

Conversely, if $f(a)=0$, then $(x-a)$ is a factor of $f(x)$.

It is not necessary to try all integer values when you are looking for possible factors. For example, with $x^3-3x^2-2x+6=0$ you need only try the factors of 6 as possible roots, i.e. ±1, ±2, ±3 and ±6.

Worked example

a Show that $x=2$ is a root of the equation $x^3-3x^2-4x+12=0$ and hence solve the equation.

b Sketch the graph of $y=x^3-3x^2-4x+12$.

Alternatively, you could factorise by long division.

Solution

a $f(2)=2^3-3(2^2)-4(2)+12=0$

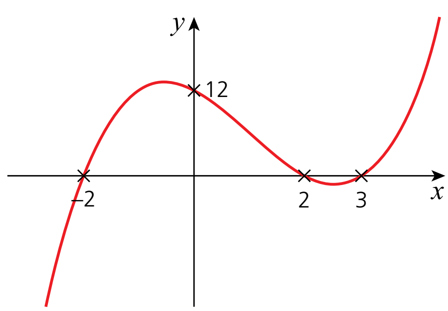
This implies that $x=2$ is a root of the equation and hence $(x-2)$ is a factor of $f(x)$.

Taking $(x-2)$as a factor gives

$$\begin{aligned}x^3-3x^2-4x+12&=(x-2)(x^2-x-6)\\&=(x-2)(x-3)(x+2)\end{aligned}$$

The solution to the equation is therefore $x=2$, $x=3$, or $x=-2$.

b The graph crosses the x-axis at $x=-2$, $x=2$ and $x=3$ and the y-axis at $y=12$.



You will not be able to factorise the expression completely in all cases, but you may be able to find one factor by inspection as in the following example.

Worked example

Given that $f(x) = x^3 - x^2 - 4x + 12$

a Show that $(x - 3)$ is a factor of $f(x)$.

b Solve the equation $f(x) = 0$.

To show that $(x-3)$ is a factor, you need to show that $f(3) = 0$.

Solution

a $f(3) = 3^3 - 3^2 - 9(3) + 9$

$= 27 - 9 - 27 + 9 = 0$

This shows that $(x - 3)$ is a factor of $f(x)$.

b Once you have found one linear factor of a cubic expression, the remaining factor is quadratic. With practice you will be able to do this step by inspection.

Check by multiplying out that you agree with this answer.

$$x^3 - x^2 - 10x + 12 = (x-3)(x^2 + 2x - 4)$$

How do you know it can't be factorised?

$x^2 + 2x - 4$ cannot be factorised so use the quadratic formula for the next step.

$$ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this example, $a = 1,\ b = 2,\ c = -4 \rightarrow x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2(-1)}$

$$\rightarrow x = \frac{-2 \pm \sqrt{20}}{-2}$$

$$\rightarrow x = 1 \pm \sqrt{5}$$

The solution to the equation is therefore $x = 3$ or $x = 1 \pm \sqrt{5}$.

Using the factor theorem to solve a cubic equation

This is very similar to earlier work except that the first step is to find a linear factor by inspection.

Worked example

a Work systematically to find a linear factor of $x^3 - 5x^2 - 2x + 24$.

b Solve the equation $x^3 - 5x^2 - 2x + 24 = 0$.

c Sketch the graph of $y = x^3 - 5x^2 - 2x + 24$.

d Sketch $y = |x^3 - 5x^2 - 2x + 24|$ on a separate set of axes.

Solution

a Let $f(x) = x^3 - 5x^2 - 2x + 24$.

Start by working systematically through all factors of 24 until you find one giving $f(x) = 0$.

$f(1) = 1 - 5 - 2 + 24 = 18$

$f(-1) = -1 - 5 + 2 + 24 = 20$

$f(2) = 8 - 20 - 4 + 24 = 8$

$f(-2) = -8 - 20 + 4 + 24 = 0$

This shows that $(x+2)$ is a linear factor of $x^3-5x^2-2x+24$.

b Factorising by inspection:

$$x^3-5x^2-2x+24=(x+2)(x^2+ax+12)$$

The second bracket starts with x^2 to get the x^3 term and finishes with 12 since $2\times 12=24$.

Looking at the x^2 term on both sides: $-5x^2=2x^2+ax^2$

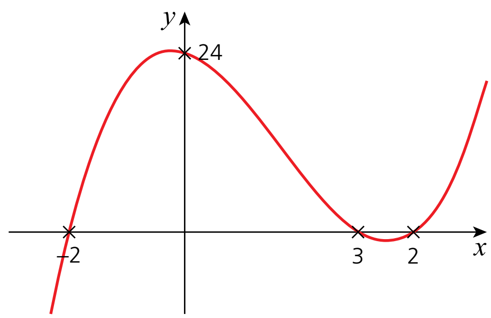
$\rightarrow a=-7$

$$x^3-5x^2-2x+24=0\rightarrow(x+2)(x^2-7x+12)=0$$

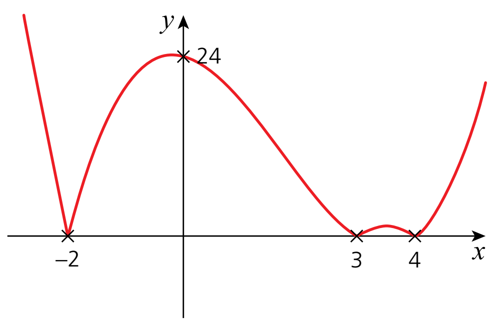
$$\rightarrow(x+2)(x-3)(x-4)=0$$

$$\rightarrow x=-2,\ x=3 \text{ or } x=4$$

c The graph is a cubic curve with a positive x^3 term that crosses the x-axis at -2, 3 and 4 and crosses the y-axis when $y=24$.



d To sketch the curve $y=|x^3-5x^2-2x+24|$, first sketch the curve as above, then reflect in the x-axis any part of the curve which is below it.



Exercise 5.2

1 Determine whether the following linear functions are factors of the given polynomials:

a x^3-8x+7; $(x-1)$ **b** x^3+8x+7; $(x+1)$

c $2x^3+3x^2-4x-1$; $(x-1)$ **d** $2x^3-3x^2+4x+1$; $(x+1)$

2 Use the factor theorem to find a linear factor of each of the following functions. Then factorise each function as a product of three linear factors and sketch its graph.

a x^3-7x-6 **b** x^3-7x+6

c x^3+5x^2-x-5 **d** x^3-5x^2-x+5

Exercise 5.2 (cont)

3 Factorise each of the following functions completely:

a x^3+x^2+x+1 **b** x^3-x^2+x-1

c x^3+3x^2+3x+2 **d** x^3-3x^2+3x-2

4 For what value of a is $(x-2)$ a factor of $x^3-2ax+4$?

5 For what value of c is $(2x+3)$ a factor of $2x^3+cx^2-4x-6$?

6 The expression x^3-6x^2+ax+b is exactly divisible by $(x-1)$ and $(x-3)$.

a Find two simultaneous equations for a and b.

b Hence find the values of a and b.

The remainder theorem

Worked example

Is there an integer root to the equation $x^3-2x^2+x+1=0$?

Solution

Since the first term is x^3 and the last term is +1, the only possible factors are $(x+1)$ and $(x-1)$.

$f(1)=1$ and $f(-1)=-3$ so there is no integer root.

This leads us to the **remainder theorem.**

Any polynomial can be divided by another polynomial of lesser order using either long division or inspection. However, there will sometimes be a remainder. The steps for algebraic long division are very similar to those for numerical long division as shown below.

Look at $(x^3-2x^2+x+1)\div(x+1)$.

Taking the first term from each (the dividend and the divisor) gives $x^3\div x=x^2$, the first term on the top in the quotient.

$-3x^2\div x$ gives $-3x$.

Similarly $4x\div x=4$

$$
\begin{array}{rl}
 & x^2-3x+4 \\
x+1 & \overline{)\,x^3-2x^2+x+1} \\
- & \underline{x^3+x^2} \\
 & \quad -3x^2+x \\
- & \quad \underline{-3x^2-3x} \\
 & \qquad\quad 4x+1 \\
- & \qquad\quad \underline{-4x+4} \\
 & \qquad\qquad -3
\end{array}
$$

This result can be written as:

$$x^3-2x^2+x+1=(x+1)(x^2-3x+4)-3.$$

Substituting $x=-1$ into both sides gives a remainder of -3.

This means that $f(-1)$ will always be the remainder when a function $f(x)$ is divided by $(x+1)$.

Generalising this gives the **remainder theorem**.

For any polynomial f(x), f(a) is the remainder when f(x) is divided by ($x - a$).

$$f(x) = (x - a)g(x) + f(a)$$

Worked example

Find the remainder when $f(x) = 2x^3 + 3x - 5$ is divided by $(x - 2)$.

Solution

Using the remainder theorem, the remainder is f(2).

$$f(2) = 2(2)^3 + 3(2) - 5 = 17$$

Worked example

When $2x^3 - 3x^2 + ax - 5$ is divided by $x - 2$, the remainder is 7. Find the value of a.

Solution

To find the remainder, substitute $x = 2$ into $2x^3 - 3x^2 + ax - 5$.

$$\begin{aligned} 2(2)^3 - 3(2)^2 + a(2) - 5 &= 7 \\ \rightarrow 16 - 12 + 2a - 5 &= 7 \\ \rightarrow -1 + 2a &= 7 \\ a &= 4 \end{aligned}$$

Exercise 5.3

1 For each function, find the remainder when it is divided by the linear factor shown in brackets:

a $x^3 + 2x^2 - 3x - 4$; $(x - 2)$

b $2x^3 + x^2 - 3x - 4$; $(x + 2)$

c $3x^3 - 3x^2 - x - 4$; $(x - 4)$

d $3x^3 + 3x^2 + x + 4$; $(x + 4)$

2 When $f(x) = x^3 + ax^2 + bx + 10$ is divided by $(x + 1)$, there is no remainder. When it is divided by $(x - 1)$, the remainder is 4. Find the values of a and b.

3 The equation $f(x) = x^3 + 4x^2 + x - 6$ has three integer roots. Solve $f(x) = 0$.

4 $(x - 2)$ is a factor of $x^3 + ax^2 + a^2x - 14$. Find all possible values of a.

5 When $x^3 + ax + b$ is divided by $(x - 1)$, the remainder is –12. When it is divided by $(x - 2)$, the remainder is also –12. Find the values of a and b and hence solve the equation $x^3 + ax + b = 0$.

6 Sketch each curve by first finding its points of intersection with the axes:

a $y = x^3 + 2x^2 - x - 2$

b $y = x^3 - 4x^2 + x + 6$

c $y = 4x - x^3$

d $y = 2 + 5x + x^2 - 2x^3$

Past–paper questions

1 The polynomial $f(x) = ax^3 - 15x^2 + bx - 2$ has a factor of $2x - 1$ and a remainder of 5 when divided by $x - 1$.

(i) Show that $b = 8$ and find the value of a. [4]

(ii) Using the values of a and b from part **(i)**, express $f(x)$ in the form $(2x - 1)\,g(x)$, where $g(x)$ is a quadratic factor to be found. [2]

(iii) Show that the equation $f(x) = 0$ has only one real root. [2]

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2 A function f is such that $f(x) = 4x^3 + 4x^2 + ax + b$. It is given that $2x - 1$ is a factor of both $f(x)$ and $f'(x)$.

(i) Show that $b = 2$ and find the value of a. [5]

Using the values of a and b from part **(i)**,

(ii) find the remainder when $f(x)$ is divided by $x + 3$, [2]

(iii) express $f(x)$ in the form $f(x) = (2x - 1)(px^2 + qx + r)$, where p, q and r are integers to be found, [2]

(iv) find the values of x for which $f(x) = 0$. [2]

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3 It is given that $f(x) = 6x^3 - 5x^2 + ax + b$ has a factor of $x + 2$ and leaves a remainder of 27 when divided by $x - 1$.

(i) Show that $b = 40$ and find the value of a. [4]

(ii) Show that $f(x) = (x + 2)(px^2 + qx + r)$, where p, q and r are integers to be found. [2]

(iii) Hence solve $f(x) = 0$. [2]

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Learning outcomes

Now you should be able to:

★ multiply two polynomials when the degree (i.e. the highest power) of at least one of them is greater than 2
★ divide one polynomial by another when the division gives no remainder
★ solve a cubic equation by first drawing the graph
★ know and use the remainder and factor theorems
★ find factors of polynomials
★ solve cubic equations.

Key points

- An expression of the form $ax^3 + bx^2 + cx + d$ where $a \neq 0$ is called a **cubic expression**.
- The graph of a cubic expression can be plotted by first calculating the value of y for each value of x in the range.
- The solution to a cubic equation is the set of values for which the corresponding graph crosses the x-axis.
- The **factor theorem** states: if $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$ and $x = a$ is a root of the equation $f(x) = 0$.
- The **remainder theorem** states: for any polynomial $f(x)$, $f(a)$ is the remainder when $f(x)$ is divided by $(x - a)$. This can be generalised to $f(x) = (x - a)g(x) + f(a)$.