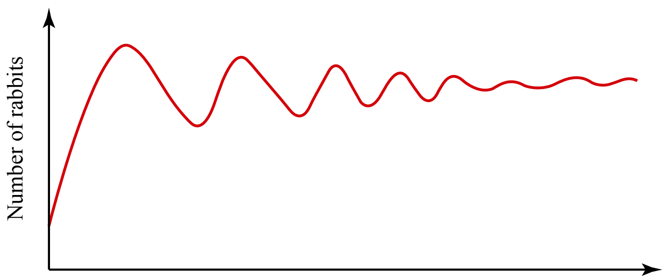
4 Rational functions and graphs

> **The purpose of visualisation is insight, not pictures.**
> *Ben Shneiderman (1947–)*

The graph in Figure 4.1 shows how the population of rabbits on a small island changes over time after a small group is introduced to the island.



▲ **Figure 4.1**

› What can you conclude from the graph?

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

4.1 Graphs of rational functions

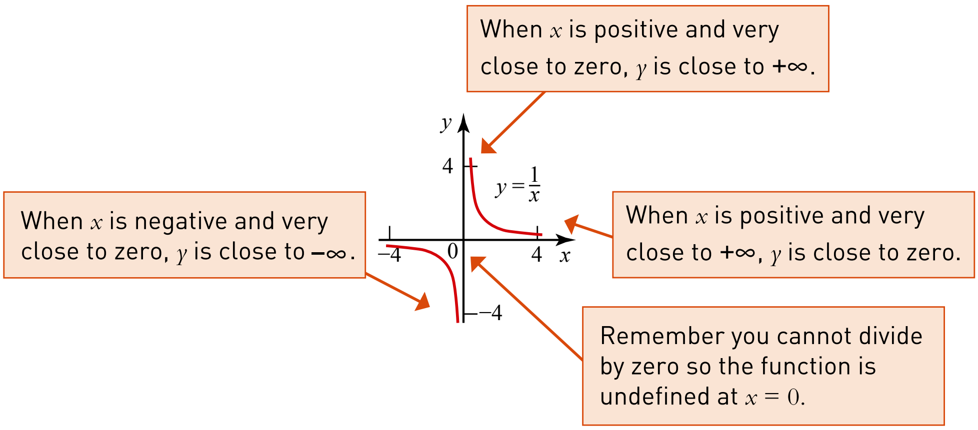
A **rational number** is defined as a number that can be expressed as $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

In a similar way, a **rational function** is defined as a function that can be expressed in the form $y = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials, and $g(x) \neq 0$.

This chapter will only look at polynomials of degree 2 (quadratics) or less.

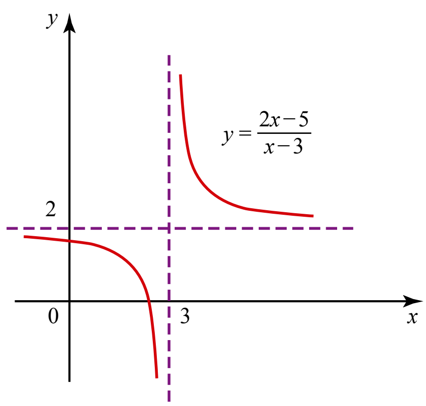
In this chapter you will learn how to sketch graphs of rational functions.

Think about the graph of $y = \frac{1}{x}$ (see Figure 4.2).



▲ **Figure 4.2**

Translating $y = \frac{1}{x}$ three units to the right and two units up gives the graph of $y = \frac{1}{x-3} + 2$ which can be written as $y = \frac{2x-5}{x-3}$ (see Figure 4.3).



▲ **Figure 4.3**

Imagine yourself moving along the curve $y = \frac{2x-5}{x-3}$ from the left. As your x coordinate gets close to 3, your y coordinate tends to $-\infty$, and you get closer and closer to the vertical line $x = 3$, shown dashed.

If you move along the curve again, letting your x coordinate increase without limit, you get closer and closer to the horizontal line $y = 2$, also shown dashed.

These dashed lines are examples of **asymptotes**. An asymptote is a straight line that a curve approaches tangentially as x and/or y tends to infinity. The line $x = 3$ is a vertical asymptote; the line $y = 2$ is a horizontal asymptote. It is usual for asymptotes to be shown by dashed lines in books. In your own work you may find it helpful to use a different colour for asymptotes.

Finding vertical asymptotes

To find any vertical asymptotes look for the values of x for which the function is undefined.

The curve $y = \frac{f(x)}{g(x)}$ is undefined when $g(x) = 0$.

Remember, you cannot divide by 0.

So there is a vertical asymptote at $x = a$ where $g(a) = 0$.

Look back at Figure 4.2 and see *Step 2* in the next section.

The signs of $f(x)$ and $g(x)$ when x is close to a, let you determine whether y tends to positive or negative infinity, as x tends to a from the left or from the right.

$y = \frac{2x - 5}{x - 3}$ is undefined when $x = 3$, so $x = 3$ is a vertical asymptote (see Figure 4.3).

> ? What are the vertical asymptotes of the graphs of the following rational functions?
>
> (i) $y = \frac{1}{x+2}$ (ii) $y = \frac{x-2}{(x-1)(x+2)}$ (iii) $y = \frac{2}{(2x-1)(x^2+1)}$

Finding horizontal asymptotes

To find any horizontal asymptotes look at:

- the value of y as $x \to \infty$
- the value of y as $x \to -\infty$.

Say 'x tends to infinity' this means as x becomes very large and positive.

When x is close to negative infinity.

For the curve $y = \frac{2x - 5}{x - 3}$, when x is numerically very large (either positive or negative) the -5 in the numerator and the -3 in the denominator become negligible compared to the value of x.

So as $x \to \pm\infty$, $y = \frac{2x - 5}{x - 3} \to \frac{2x}{x} = 2$.

Hence the line $y = 2$ is a horizontal asymptote (see Figure 4.3).

Think about the value of y when x is:

- large and positive, e.g. +1000 and +10 000
- 'large' and negative, e.g. −1000 and −10 000.

> ? What are the horizontal asymptotes of the graphs of the following rational functions?
>
> (i) $y = \frac{1}{x+2}$ (ii) $y = \frac{x}{x+2}$ (iii) $y = \frac{1-2x}{x+2}$

Technology note

If you have graphing software, you can use it to sketch graphs and check that you have found the asymptotes correctly.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

4.2 How to sketch a graph of a rational function

The five steps below show how to draw a sketch graph of $y = \frac{(x+2)}{(x-2)(x+1)}$.

Step 1: Find where the graph cuts the axes

When $x = 0$:

$$y = \frac{(x+2)}{(x-2)(x+1)} = \frac{2}{-2 \times 1} = -1$$

$\Rightarrow$ the y intercept is at $(0,-1)$

When $y = 0$:

$$\frac{(x+2)}{(x-2)(x+1)} = 0$$

$\Rightarrow x + 2 = 0$

$\Rightarrow x = -2$

$\Rightarrow$ the x intercept is at $(-2,0)$

Step 2: Find the vertical asymptotes and examine the behaviour of the graph either side of them

$y = \frac{(x+2)}{(x-2)(x+1)}$ has vertical asymptotes when $(x-2)(x+1) = 0$

$\Rightarrow$ the vertical asymptotes are at $x = -1$ and $x = 2$

On either side of the asymptote, y will either be large and positive (tending to $+\infty$) or large and negative (tending to $-\infty$). To find out which, you need to examine the sign of y on either side of the asymptote.

Behaviour of the graph $y = \frac{(x+2)}{(x-2)(x+1)}$ either side of the asymptote $x = 2$:

When x is slightly less than 2 then

$$y \text{ is } \frac{(+ve\ number)}{(-ve\ number\ close\ to\ zero)(+ve\ number)}.$$

So y is large and negative.

For example, when $x = 1.999$

$$y = \frac{(1.999+2)}{(1.999-2)(1.999+1)} = \frac{3.999}{-0.001 \times 2.999} = -1333.44\ldots$$

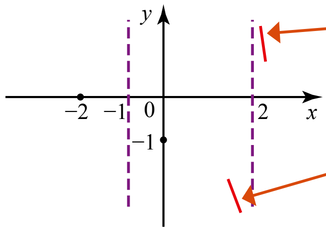
When x is slightly more than 2 then

$$y \text{ is } \frac{(+ve\ number)}{(+ve\ number\ close\ to\ zero)(+ve\ number)}.$$

So y is large and positive.

For example, when $x = 2.001$

$$y = \frac{(2.001+2)}{(2.001-2)(2.001+1)} = \frac{4.001}{0.001 \times 3.001} = 1333.22\ldots$$



$y \to +\infty$ as $x \to 2$ from the right.

$y \to -\infty$ as $x \to 2$ from the left.

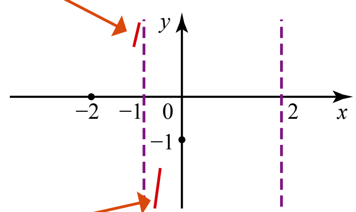
▲ **Figure 4.4**

Behaviour of the graph $y = \dfrac{(x+2)}{(x-2)(x+1)}$ either side of the asymptote $x = -1$:

When x is slightly less than −1 then:

$$y \text{ is } \frac{(+ve\ number)}{(-ve\ number)(-ve\ number\ close\ to\ zero)}$$

so y is large and positive.



When x is slightly more than −1 then:

$$y \text{ is } \frac{(+ve\ number)}{(-ve\ number)(+ve\ number\ close\ to\ zero)}$$

so y is large and negative.

▲ **Figure 4.5**

Step 3: Find the horizontal asymptotes and examine the behaviour of the graph either side of them

Examine the behaviour as x tends to infinity.

As $x \to \pm\infty$, $y = \dfrac{(x+2)}{(x-2)(x+1)} \to \dfrac{x}{x^2} = \dfrac{1}{x} \to 0$

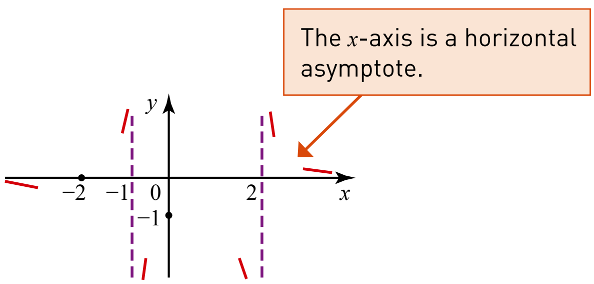
This means that the line $y = 0$ is a horizontal asymptote.

When x is very large (either positive or negative) the 2 in the numerator and the −2 and the 1 in the denominator become negligible compared to the vaxlues of x, so you can ignore them.

> What are the signs of $(x+2)$, $(x-2)$ and $(x+1)$ for:
(i) large, positive values of x
(ii) large, negative values of x?

> What is the sign of $y = \dfrac{(x+2)}{(x-2)(x+1)}$ for:
(iii) large, positive values of x
(iv) large, negative values of x?

From the discussion point above, you now know that $y \to 0$ from above as $x \to \infty$, and $y \to 0$ from below as $x \to -\infty$. This additional information is shown in Figure 4.6.



▲ **Figure 4.6**

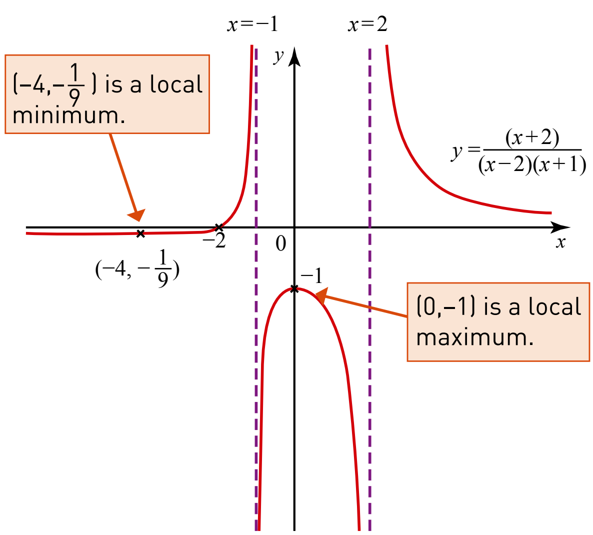
Step 4: Identify any stationary points

At a stationary point, $\dfrac{dy}{dx} = 0$.

> Show that $y = \dfrac{(x+2)}{(x-2)(x+1)}$ has stationary points at $(-4, -\frac{1}{9})$ and $(0, -1)$.

Step 5: Complete the sketch

The sketch is completed in Figure 4.7.



▲ **Figure 4.7**

Notice that, in this case, you can conclude that there is a local minimum at $(-4,-\frac{1}{9})$ and a local maximum at $(0,-1)$ without the need for further differentiation.

So the range of the function $f(x) = \frac{(x+2)}{(x-2)(x+1)}$ is $f(x) \geq -\frac{1}{9}$ and $f(x) \leqslant -1$.

You can use the sketch to solve inequalities.

For example: $\frac{(x+2)}{(x-2)(x+1)} \leq 0$ when $x \leqslant -2$ or $-1 < x < 2$.

For these values of x, the curve is below the x-axis.

Using the discriminant to find the range of a function

You can use the discriminant to find the range of the function instead of using differentiation. This method is often more straightforward than differentiation.

Look at Figure 4.7, this graph meets the horizontal line $y = k$ where $k = \frac{x+2}{(x-2)(x+1)}$. The range of the function $f(x) = \frac{x+2}{(x-2)(x+1)}$ will be the values of k for which this equation has real roots.

$$k = \frac{x+2}{(x-2)(x+1)}$$

$$\Rightarrow \quad k(x-2)(x+1) = x+2$$

$$\Rightarrow \quad kx^2 - kx - 2k = x+2$$

$$\Rightarrow \quad kx^2 - (k+1)x - 2k - 2 = 0$$

This equation has real roots when the discriminant, $b^2 - 4ac$, is positive or zero.

So $(k+1)^2 - 4k(-2k-2) \geqslant 0$

$$k^2 + 2k + 1 + 8k^2 + 8k \geqslant 0$$

$$9k^2 + 10k + 1 \geqslant 0$$

$$(k+1)(9k+1) \geqslant 0$$

$$\Rightarrow k \leqslant -1 \text{ or } k \geqslant -\frac{1}{9}$$

So the function $f(x) = \frac{x+2}{(x-2)(x+1)}$ cannot take any values between -1 and $-\frac{1}{9}$, and the range of the function is $f(x) \leqslant -1$ and $f(x) \geqslant -\frac{1}{9}$ as found before.

Using symmetry

Recognising symmetry can help you to draw a sketch.

- If $f(x) = f(-x)$ the graph of $y = f(x)$ is symmetrical about the y-axis.
- If $f(x) = -f(-x)$ the graph of $y = f(x)$ has rotational symmetry of order 2 about the origin.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

Note

A function is an **even function** if its graph has the y-axis as a line of symmetry.

So when $f(x) = f(-x)$ the function is **even**.

A function is an **odd function** if its graph has rotational symmetry of order 2 about the origin.

So when $f(x) = -f(-x)$ the function is **odd**.

Example 4.1

(i) Sketch the graph of $y = f(x)$, where $f(x) = \dfrac{x^2+1}{x^2+2}$.

(ii) State the range of $f(x)$.

(iii) The equation $f(x) = k$ has no real solutions.

Find the values of k.

Solution

(i) *Step 1:*

When $x = 0$, $y = \frac{1}{2}$, so the graph passes through $\left(0, \frac{1}{2}\right)$.

No (real) value of x makes $x^2 + 1 = 0$, so the graph does not cut the x-axis.

Step 2:

No (real) value of x makes $x^2 + 2 = 0$, so there are no vertical asymptotes.

Step 3:

As $x \to \pm\infty$, $y = \dfrac{x^2+1}{x^2+2} \to \dfrac{x^2}{x^2} = 1$

So $y = 1$ is a horizontal asymptote.

The denominator is larger than the numerator for all values of x, so $y < 1$ for all x.

So $y \to 1$ from below as $x \to \pm\infty$.

Step 4:

Differentiate to find the stationary points.

$$y = \frac{x^2+1}{x^2+2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)\times 2x - (x^2+1)\times 2x}{(x^2+2)^2}$$

$$= \frac{2x^3 + 4x - 2x^3 - 2x}{(x^2+2)^2}$$

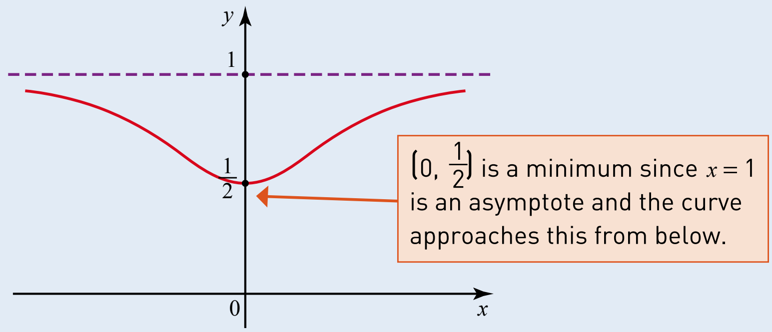
$$= \frac{2x}{(x^2+2)^2}$$

At a stationary point, $\frac{dy}{dx} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$.

When $x = 0$ then $y = \frac{1}{2}$, so there is a stationary point at $\left(0, \frac{1}{2}\right)$.

Step 5:

$f(x)$ contains only even powers of x, so $f(x) = f(-x)$ and the graph is symmetrical about the y-axis (see Figure 4.8).



▲ **Figure 4.8**

(ii) From the sketch the range is $\frac{1}{2} \leqslant f(x) < 1$.

(iii) Solutions of the equation $f(x) = k$ occur where the horizontal line $y = k$ meets the curve $y = f(x)$.

From the sketch of $y = f(x)$, you can see that when $k < \frac{1}{2}$ or $k \geqslant 1$, the line $y = k$ will not meet the curve and so there are no solutions to the equation $f(x) = k$.

Show that you get the same answer when you use the discriminant method to find the range.

Exercise 4A

Follow the steps below for each of questions **1** to **12**.

Step 1: Find the coordinates of the point(s) where the graph cuts the axes.
Step 2: Find the vertical asymptote(s).
Step 3: State the behaviour of the graph as $x \to \pm\infty$.
Step 4: Sketch the graph.
Step 5: Find the range.

1 $y = \frac{2}{x-3}$

2 $y = \frac{2}{(x-3)^2}$

3 $y = \frac{1}{x^2+1}$

4 $y = \frac{x}{x^2-4}$

5 $y = \frac{2-x}{x+3}$

6 $y = \frac{x-5}{(x+2)(x-3)}$

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

7 $y = \dfrac{3-x}{(2-x)(4-x)}$

8 $y = \dfrac{x}{x^2+3}$

9 $y = \dfrac{x-3}{(x-4)^2}$

10 $y = \dfrac{(2x-3)(5x+2)}{(x+1)(x-4)}$

11 $y = \dfrac{x^2-6x+9}{x^2+1}$

12 $y = \dfrac{x^2-5x-6}{(x+1)(x-4)}$

PS **13** (i) Sketch the graph of $y = \dfrac{4-x^2}{4+x^2}$.

(ii) The equation $\dfrac{4-x^2}{4+x^2} = k$ has no real solutions.

Find the possible values of k.

PS **14** (i) Sketch the graph of $y = \dfrac{1}{(x+1)(3-x)}$.

(ii) Write down the equation of the line of symmetry of the graph and hence find the coordinates of the local minimum point.

(iii) For what values of k does the equation $\dfrac{1}{(x+1)(3-x)} = k$ have

(a) two real distinct solutions

(b) one real solution

(c) no real solutions?

PS **15** Solve these inequalities, by first sketching one or more appropriate curve(s).

(i) $\dfrac{x+2}{x-1} \geqslant 0$

(ii) $\dfrac{2x+3}{x-2} \leqslant 1$

(iii) $\dfrac{x-5}{x+1} \leqslant \dfrac{1}{x-3}$

(iv) $\dfrac{x+3}{2x-1} \geqslant 2$

(v) $\dfrac{2x-1}{x+3} \leqslant \dfrac{1}{2}$

(vi) $\dfrac{1}{x+6} \leqslant \dfrac{2}{2-3x}$

4.3 Oblique asymptotes

In general, when the numerator of any rational function is of lower degree than the denominator (e.g. $y = \dfrac{(x+2)}{(x-2)(x+1)}$), then $y = 0$ is a horizontal asymptote.

When the numerator has the same degree as the denominator (e.g. $y = \dfrac{2x-5}{x-3}$), then as $x \to \pm\infty$, y tends to a fixed rational number. So there is a horizontal asymptote of the form $y = c$.

When the degree of the numerator is one greater than that of the denominator (e.g. $y = \frac{2x^2 - 4x - 1}{x - 3}$), then as $x \to \pm\infty$ then y tends to an expression in the form $ax + b$. So the asymptote is a sloping line, you say there is an **oblique asymptote**.

To find the equation of the oblique asymptote of $y = \frac{2x^2 - 4x - 1}{x - 3}$ you need to rewrite the equation using long division.

$$\begin{array}{r}
2x \quad + 2 \\
(x-3)\overline{)2x^2 - 4x - 1} \\
-\ \underline{2x^2 - 6x} \quad\quad \\
2x - 1 \\
-\ \underline{2x - 6} \\
5
\end{array}$$

So $\qquad y = \frac{2x^2 - 4x - 1}{x - 3} = 2x + 2 + \frac{5}{x - 3}$

As x increases then $\frac{5}{x-3} \to 0$ and $y \to 2x + 2$, so the equation of the oblique asymptote is $y = 2x + 2$.

Notice you didn't need to complete the division.

In order to sketch the graph of $y = \frac{2x^2 - 4x - 1}{x - 3}$ you need to consider any vertical asymptotes and then examine the behaviour of the function on either side of the asymptotes.

The vertical asymptote is at $x = 3$.

You cannot divide by 0, so $x = 3$ is an asymptote.

Examine the behaviour of $y = \frac{2x^2 - 4x - 1}{x - 3}$ on either side of the vertical asymptote.

- When x is slightly less than 3 then $y \to -\infty$.
- When x is slightly more than 3 then $y \to +\infty$.

When x is slightly less than 3 then $y = \frac{+ve}{-ve} = -ve$

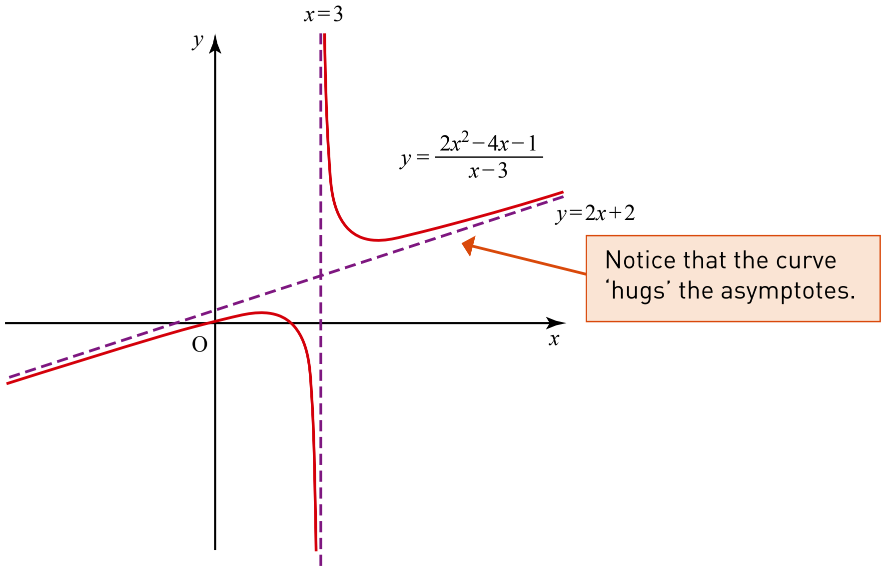
When x is slightly more than 3 then $y = \frac{+ve}{+ve} = +ve$

Examine the behaviour on either side of the oblique asymptote:

- When x is slightly more than 3 then $y = 2x + 2 + \frac{5}{x-3} > 2x + 2$ so the curve lies above the asymptote.
- When x is slightly less than 3 then $y = 2x + 2 + \frac{5}{x-3} < 2x + 2$ so the curve lies below the asymptote.

For $x > 3$, $\frac{5}{x-3}$ is positive so $y > 2x + 2$.

For $x < 3$, $\frac{5}{x-3}$ is negative so $y < 2x + 2$.



▲ **Figure 4.9**

Note

Rewriting the equation as $y = 2x + 2 + \dfrac{5}{x-3}$ makes it easier to differentiate and find the turning points.

$$y = 2x + 2 + \frac{5}{x-3}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{5}{(x-3)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 2 - \frac{5}{(x-3)^2} = 0$$

So $\dfrac{5}{(x-3)^2} = 2 \Rightarrow (x-3)^2 = \dfrac{5}{2}$

$$\Rightarrow \quad (x-3) = \pm\sqrt{\frac{5}{2}}$$

$$\Rightarrow \quad x = 3 \pm \sqrt{\frac{5}{2}}$$

So there are turning points at $x = 3 \pm \sqrt{\dfrac{5}{2}}$.

The turning points are at $x = 1.42$ and $x = 4.58$ (to 3 significant figures).

Exercise 4B

For each of the curves given in questions **1** to **3**:

(a) find the coordinates of any point(s) where the graph cuts the axes

(b) find the equations of the asymptotes

(c) find the coordinates of the turning points

(d) sketch the graph.

1 (i) $y = 2x - 1 + \dfrac{2}{x - 4}$ (ii) $y = 2x - 1 + \dfrac{2}{4 - x}$

2 (i) $y = \dfrac{x^2 - 4x + 6}{x - 1}$ (ii) $y = \dfrac{x^2 - 4x + 6}{1 - x}$

3 (i) $y = \dfrac{(2x - 1)(x - 4)}{(x - 3)}$ (ii) $y = \dfrac{(2x - 1)(x - 4)}{(3 - x)}$

4 A curve C has equation $y = \dfrac{2x^2 + x - 1}{x - 1}$. Find the equations of the asymptotes of C.

Show that there is no point on C for which $1 < y < 9$.

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5 The curve C has equation $y = \dfrac{x^2 + px + 1}{x - 2}$, where p is a constant. Given that C has two asymptotes, find the equation of each asymptote.

Find the set of values of p for which C has two distinct turning points.

Sketch C in the case $p = -1$. Your sketch should indicate the coordinates of any intersections with the axes, but need not show the coordinates of any turning points.

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6 The curve C has equation $y = \dfrac{2x^2 + kx}{x + 1}$, where k is a constant. Find the set of values of k for which C has no stationary points.

For the case $k = 4$, find the equations of the asymptotes of C and sketch C, indicating the coordinates of the points where C intersects the coordinate axes.

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7 The curve C has equation

$$y = \frac{px^2 + 4x + 1}{x + 1},$$

where p is a positive constant and $p \neq 3$.

(i) Obtain the equations of the asymptotes of C.

(ii) Find the value of p for which the x-axis is a tangent to C, and sketch C in this case.

(iii) For the case $p = 1$, show that C has no turning points, and sketch C, giving the exact coordinates of the points of intersection of C with the x-axis.

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8 The curve C has equation

$$y = \lambda x + \frac{x}{x-2},$$

where λ is a non-zero constant. Find the equations of the asymptotes of C.

Show that C has no turning points if $\lambda < 0$.

Sketch C in the case $\lambda = -1$, stating the coordinates of the intersections with the axes.

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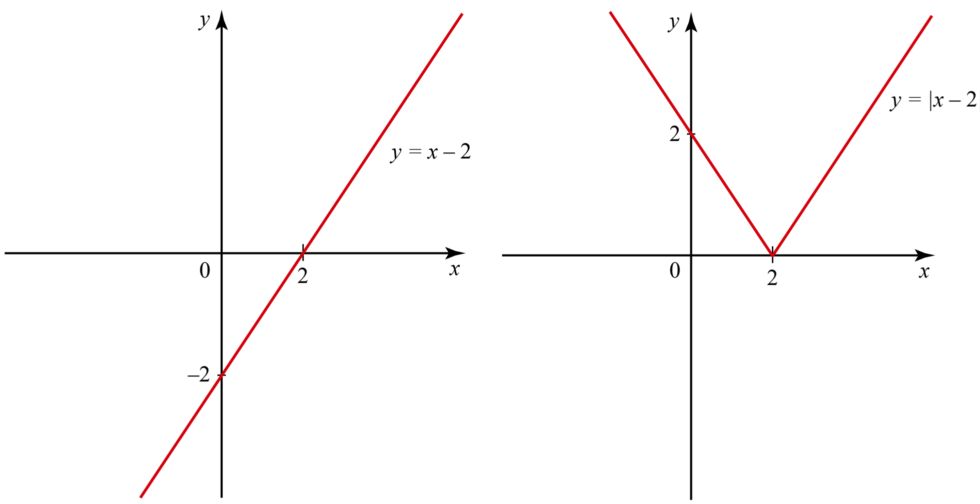
4.4 Sketching curves related to $y = f(x)$

The curve $y = |f(x)|$

The function $|f(x)|$ is the **modulus** of $f(x)$. $|f(x)|$ always takes the positive numerical value of $f(x)$. For example, when $f(x) = -2$, then $|f(x)| = 2$.



▲ **Figure 4.10 (i)** $y = x - 2$ **(ii)** $y = |x - 2|$

The graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by replacing values where $f(x)$ is negative by the equivalent positive values. This is the same as reflecting that part of the curve in the x-axis.

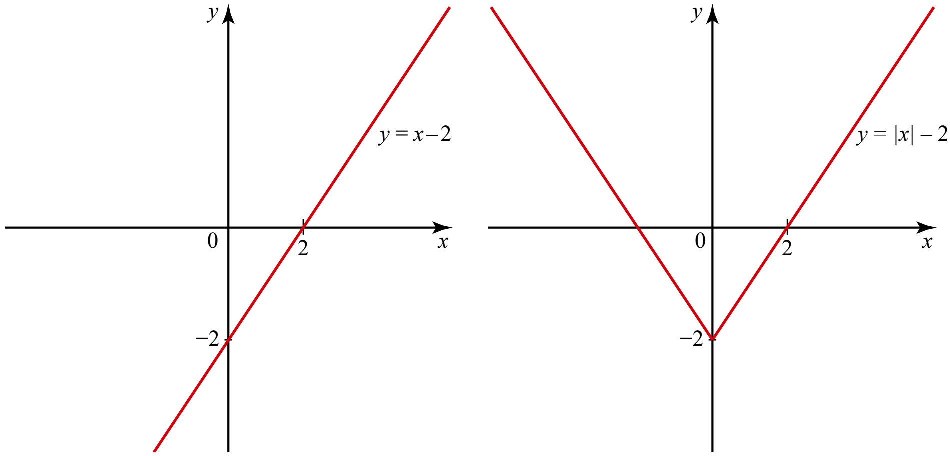
To draw the curve $y = |f(x)|$ you should:

- draw $y = f(x)$
- reflect the part(s) of the curve where $y < 0$ in the x-axis.

The curve $y = |f(x)|$

Remember that $|x|$ always takes the positive numerical value of x.

When $x = 2$ then $f(|2|) = f(2)$, and when $x = -2$ then $f(|-2|) = f(2)$

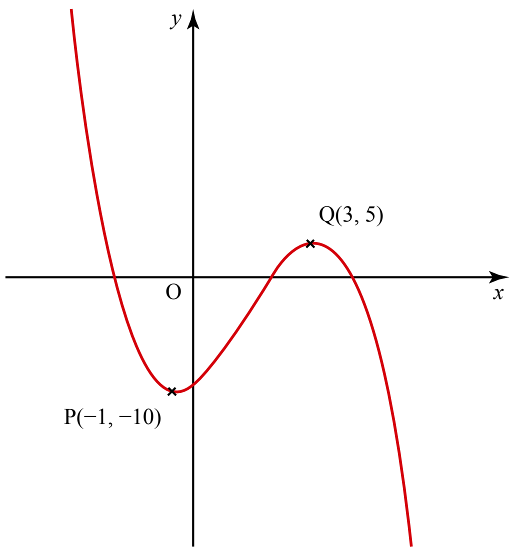


▲ **Figure 4.11** **(i)** $y = x - 2$ **(ii)** $y = |x| - 2$

To draw the curve $y = f(|x|)$ you should:

- draw $y = f(x)$ for $x > 0$
- reflect the part of the curve where $x > 0$ in the y axis.

Example 4.2



▲ **Figure 4.12**

The diagram shows the curve $y = f(x)$. The curve has stationary points at P and Q.

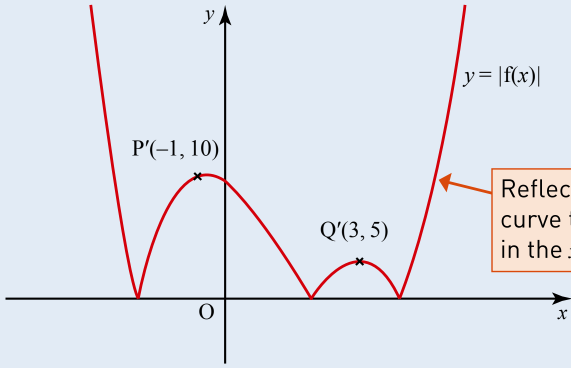
Sketch the curves

(i) $y = |f(x)|$ (ii) $y = f(|x|)$.

➜

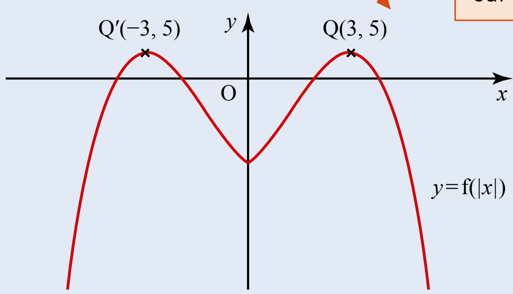
Solution

(i)



▲ **Figure 4.13**

(ii)



▲ **Figure 4.14**

The curve $y = \frac{1}{f(x)}$

You can use the following points to help you sketch $y = \frac{1}{f(x)}$ given the graph of $y = f(x)$.

» When $f(x) = 1$, then $\frac{1}{f(x)} = 1$.

Also when $f(x) = -1$, then $\frac{1}{f(x)} = -1$.

So the curves $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect when $y = 1$ and $y = -1$.

The points $(x, 1)$ and $(x, -1)$ are fixed points; they are the same on both curves.

» $f(x)$ and $\frac{1}{f(x)}$ have the same sign.

When $f(x)$ is above the x-axis then so is $\frac{1}{f(x)}$. Similarly, when $f(x)$ is below the x-axis then so is $\frac{1}{f(x)}$.

» $\frac{1}{f(x)}$ is undefined when $f(x) = 0$.

So $y = \frac{1}{f(x)}$ has a discontinuity at any point where $f(x) = 0$ (i.e. the x intercept). This is usually an asymptote.
For example, if $x = 3$ is a root of $y = f(x)$ then $x = 3$ is an asymptote of $\frac{1}{f(x)}$.

Roots become asymptotes.

Asymptotes become discontinuities. You should show these 'apparent roots' with a small open circle on the x-axis.

» If f(x) has a vertical asymptote at $x = a$, then $f(x) \to \pm\infty$ as $x \to a$.

As $f(x) \to \infty$, $\frac{1}{f(x)} \to 0$ so $\frac{1}{f(x)}$ approaches the x-axis as $x \to a$.

However, since f(a) is not defined, $\frac{1}{f(a)}$ is not defined either, so $y = \frac{1}{f(x)}$ has a discontinuity at $x = a$.

For example, if $x = 3$ is an asymptote of $y = f(x)$ then $x = 3$ is a discontinuity of $\frac{1}{f(x)}$.

The gradient of $\frac{1}{f(x)}$ at a given point has the opposite sign to f(x).

» $$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{[f(x)]^2}$$

Hence:

- when f(x) is increasing, $\frac{1}{f(x)}$ is decreasing
- when f(x) is decreasing, $\frac{1}{f(x)}$ is increasing
- when $f'(x) = 0$ then $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = 0$.

So where f(x) is a maximum, $\frac{1}{f(x)}$ is a minimum...

... and where f(x) is a minimum, $\frac{1}{f(x)}$ is a maximum

?

› Prove that $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{[f(x)]^2}$.

Example 4.3

Given

$$f(x) = \cos x$$

and $$g(x) = \tan x, x \neq \pm 90, \pm 270,$$

sketch the graphs of

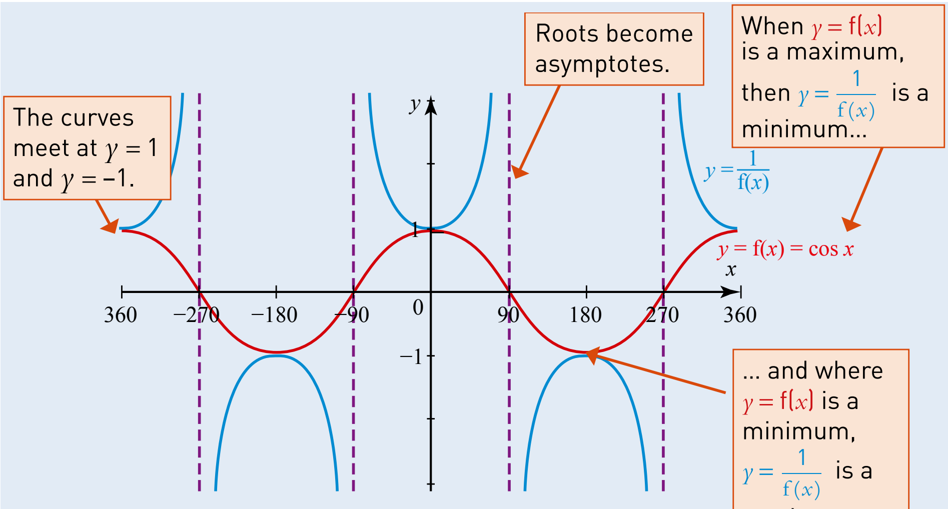
(i) $y = \frac{1}{f(x)}$ (ii) $y = \frac{1}{g(x)}$

for $-360° \leqslant x \leqslant 360°$.

Solution

(i) Start by drawing the graph of $y = \cos x$. This is the curve shown in red in Figure 4.15.

→

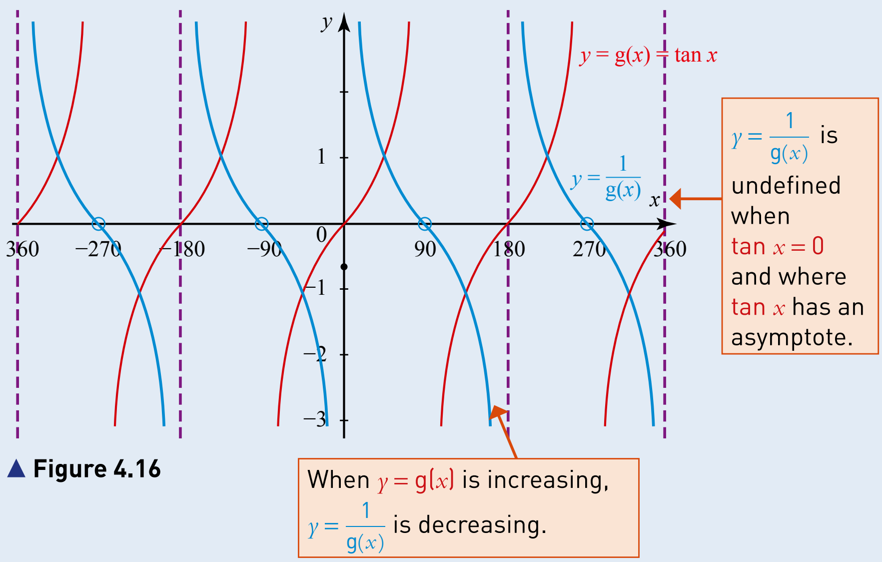


▲ **Figure 4.15**

(ii) Start by drawing the graph of $y = \tan x$. This is the curve shown in red in Figure 4.16.

Note the asymptotes for $y = \tan \theta$ have been omitted for clarity.

Notice $y = f(x)$ and $y = \frac{1}{f(x)}$ meet at the points where $y = 1$ and when $y = -1$



▲ **Figure 4.16**

Note

You met the graphs of the reciprocal trigonometric functions in *Pure Mathematics 2 and 3*.

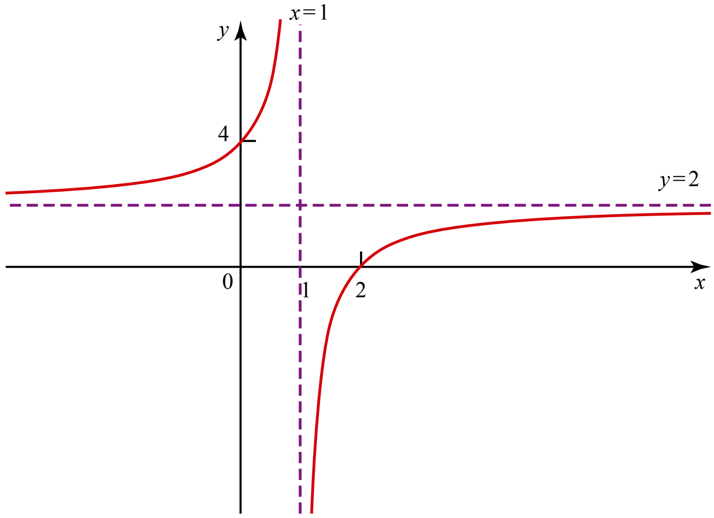
In part (ii) of Example 4.3, the points where $x = \pm 90, \pm 270$ were excluded because the function $g(x)$ was not defined for $x = \pm 90, \pm 270$ and so $\frac{1}{g(x)}$ is also not defined for these values this is shown by the small open circles on the x-axis.

Note when you draw $y = \cot x$, then there will be roots at $x = \pm 90, \pm 270$ ... (where $\cos x = 0$) and asymptotes where $\sin x = 0$ since $\cot x = \frac{\cos x}{\sin x}$.

Example 4.4

The diagram shows the curve $y = f(x)$. The lines $x = 1$ and $y = 2$ are asymptotes to the curve and the curve intercepts the axes at (0, 4) and (2, 0).

Sketch the curve $y = \frac{1}{f(x)}$.



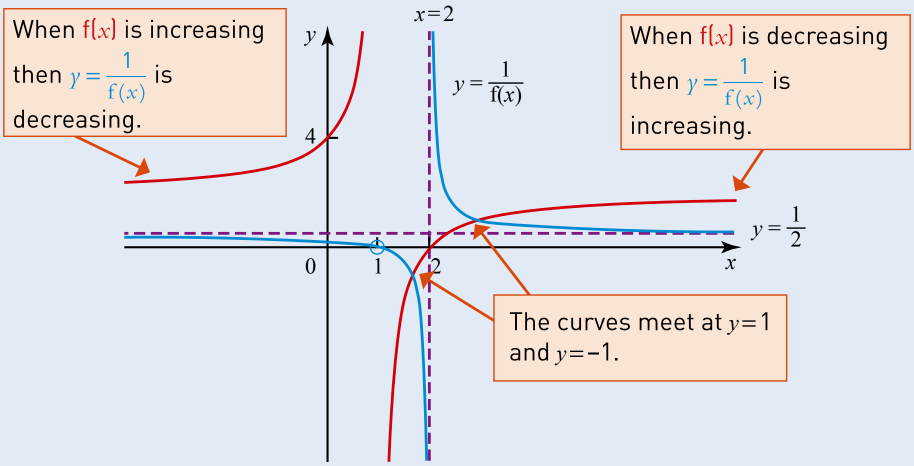
▲ **Figure 4.17**

Solution

From the graph you can see that:

- $x = 2$ is a root of $y = f(x) \Rightarrow x = 2$ is an asymptote of $y = \frac{1}{f(x)}$.
- $y = 2$ is an asymptote of $y = f(x) \Rightarrow y = \frac{1}{2}$ is an asymptote of $y = \frac{1}{f(x)}$.
- $x = 1$ is an asymptote of $y = f(x) \Rightarrow x = 1$ is a discontinuity of $y = \frac{1}{f(x)}$.

Use an open circle to show that $x = 1$ is not part of the curve.



▲ **Figure 4.18**

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? Let $f(x) = \dfrac{x-4}{x-1}, x \neq 1$ and $g(x) = \dfrac{x-1}{x-4}, x \neq 4$.

Are the graphs of $y = \dfrac{1}{f(x)}$ and $y = g(x)$ the same? Explain your answer fully.

The curve $y^2 = f(x)$

You can think of $y^2 = f(x)$ as two curves:

$$y = \sqrt{f(x)} \text{ and } y = -\sqrt{f(x)}$$

$y = -\sqrt{f(x)}$ is a reflection of $y = \sqrt{f(x)}$ in the x-axis.

You can use the following points to help you sketch $y^2 = f(x)$ given the graph of $y = f(x)$.

- $y^2 = f(x)$ is symmetrical about the x-axis.
- $y^2 = f(x)$ is undefined where $f(x) < 0$.
  So any parts of $y = f(x)$ that are below the x-axis will not be part of $y^2 = f(x)$.

Since the square root of a negative number is not real.

- $y = f(x)$ and $y^2 = f(x)$ intersect where $y = 0$ or $y = 1$.

$0^2 = 0$ and $1^2 = 1$.

- $y^2 = f(x) \Rightarrow 2y\dfrac{dy}{dx} = f'(x) \Rightarrow \dfrac{dy}{dx} = \dfrac{f'(x)}{2y}$.
  So when $y > 0$, the gradients of $y = f(x)$ and $y^2 = f(x)$ have the same sign for a given value of x.
  Also $y = f(x)$ and $y^2 = f(x)$ have stationary points located at the same x values.

So when $y = f(x)$ is increasing so is $y^2 = f(x)$…

…and when $y = f(x)$ is decreasing so is $y^2 = f(x)$.

- When $y = f(x)$ has a root then, provided $f'(x) \neq 0$, $y^2 = f(x)$ passes vertically through the x-axis.

Provided the root is not at a stationary point.

Example 4.5

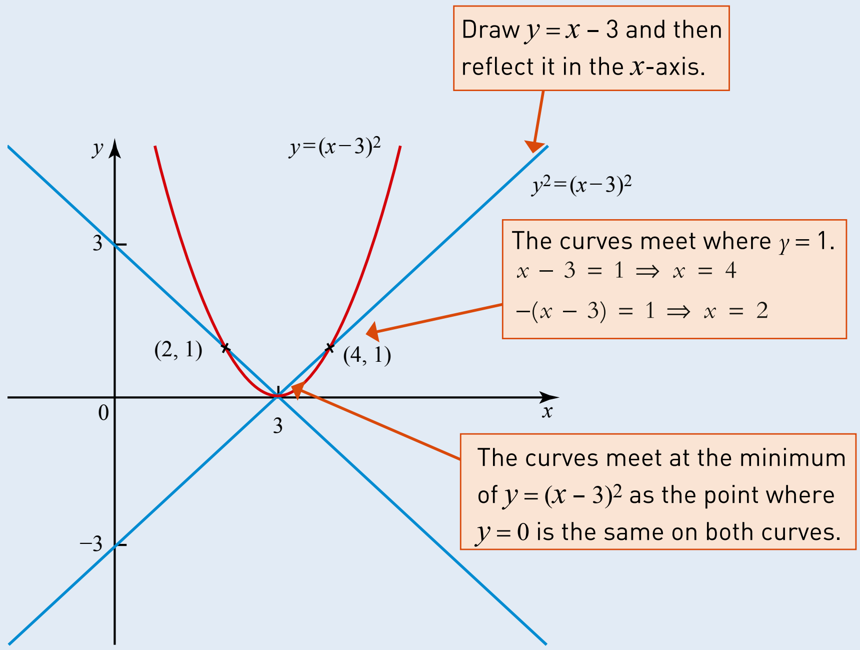
(i) Given $f(x) = (x-3)^2$, sketch $y = f(x)$ and $y^2 = f(x)$ on the same axes.

(ii) Given $f(x) = \cos x$ for $-2\pi \leqslant x \leqslant 2\pi$, sketch $y = f(x)$ and $y^2 = f(x)$ on the same axes.

Solution

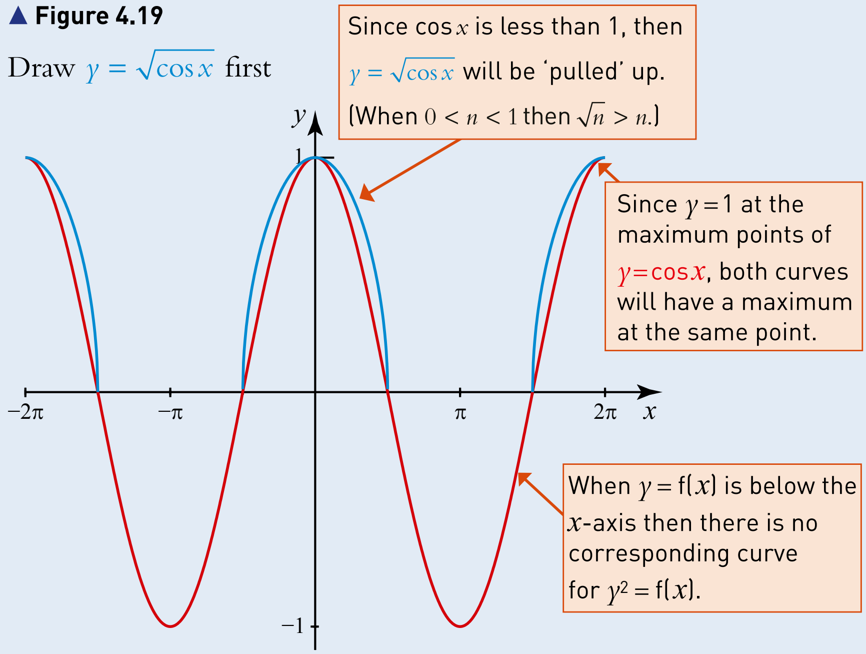
(i) $y = (x-3)^2$ is a translation of $y = x^2$ by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

$y^2 = (x-3)^2 \Rightarrow y = \pm(x-3)$



▲ **Figure 4.19**

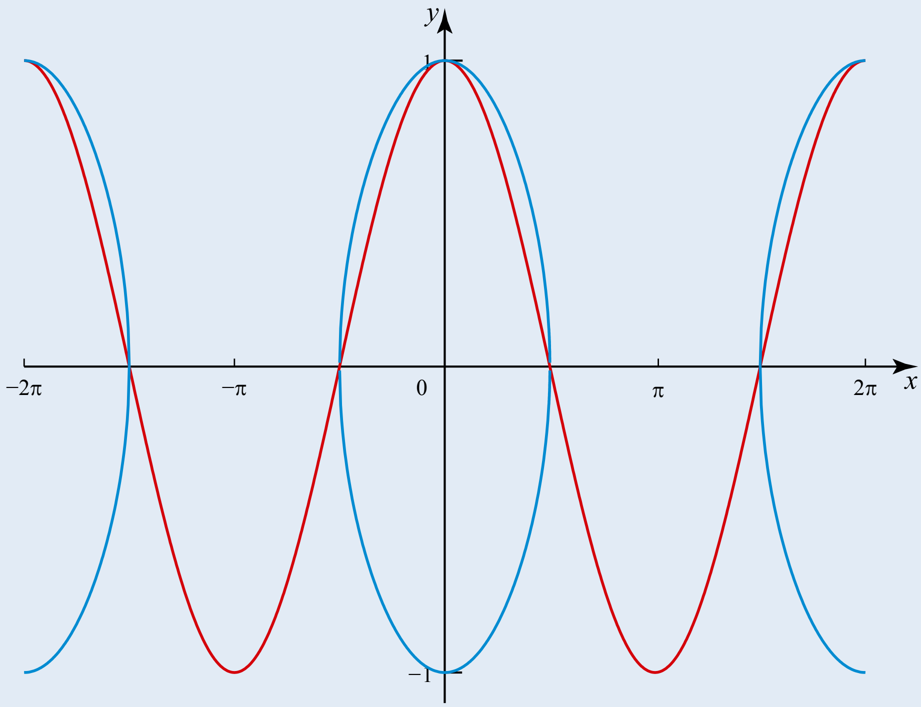
(ii) Draw $y = \sqrt{\cos x}$ first



▲ **Figure 4.20**

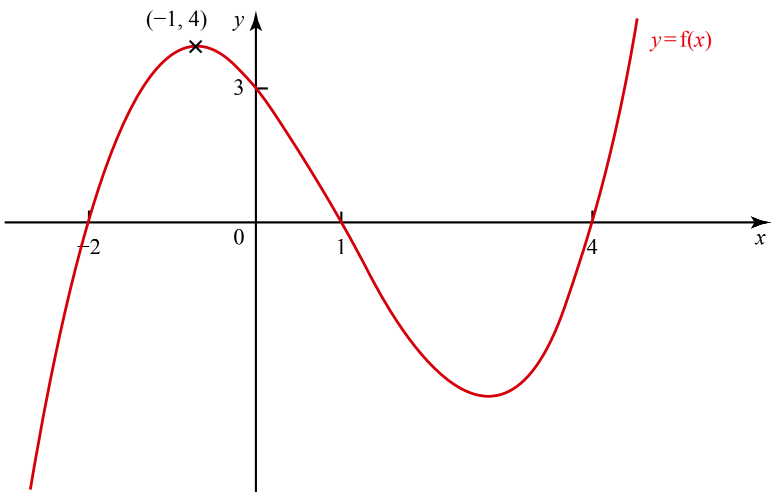
Reflect the curve $y = \sqrt{\cos x}$ in the x-axis to give $y = \pm\sqrt{\cos x}$.

➜



▲ **Figure 4.21**

Example 4.6



▲ **Figure 4.22**

The diagram shows the curve $y = \mathrm{f}(x)$.

The curve has a maximum point at $(-1, 4)$. It crosses the x-axis at $(-2, 0)$, $(1, 0)$ and $(3, 0)$ and the y-axis at $(0, 3)$.

Sketch the curve $y^2 = \mathrm{f}(x)$, showing the coordinates of any turning points and where the curve crosses the axes.

Square root the y coordinate to find the corresponding point on $y = \sqrt{f(x)}$.

Solution

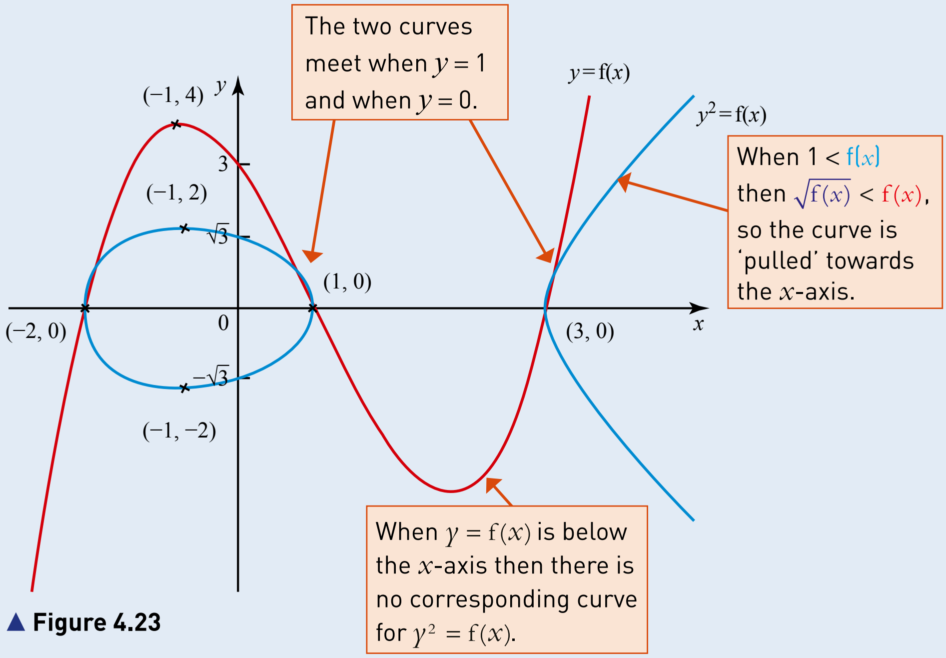
Think about the curve $y = \sqrt{f(x)}$ first.

$y = f(x)$ has a maximum at $(-1, 4)$ so $y = \sqrt{f(x)}$ has a maximum at $(-1, 2)$.

$y = f(x)$ has a y intercept at $(0, 3)$ so $y = \sqrt{f(x)}$ has a y intercept at $(0, \sqrt{3})$

The x intercepts remain the same.

Draw $y = \sqrt{f(x)}$ first and then reflect the curve in the x-axis to complete the sketch.



▲ **Figure 4.23**

Exercise 4C

For each of questions **1** to **5**:

Sketch the following curves, showing the coordinates of any turning points and where the curve crosses the axes. Label any asymptotes with their equations.

(i) $y = f(x)$

(ii) $y = |f(x)|$

(iii) $y = f(|x|)$

(iv) $y = \dfrac{1}{f(x)}$

(v) $y^2 = f(x)$

1 $f(x) = x^2 - 4$

2 $f(x) = \sin x$, for $-2\pi \leqslant x \leqslant 2\pi$.

3 $f(x) = e^x$

4 $f(x) = (2 - x)(x + 1)(2x - 1)$

5 $f(x) = \dfrac{x + 3}{x + 1}, \quad x \neq -1$

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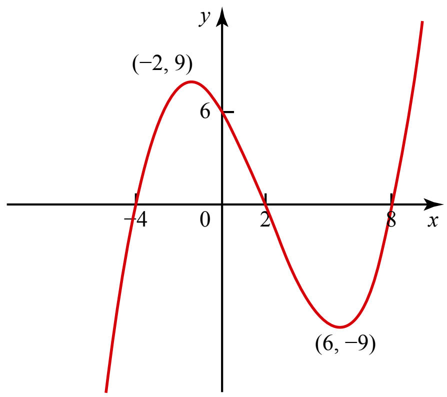
6



The diagram shows the curve $y = f(x)$.

The curve has turning points at $(-2, 9)$ and $(6, -9)$. It crosses the x-axis at $(-4, 0)$, $(2, 0)$ and $(8, 0)$ and the y-axis at $(0, 6)$.

Sketch the following curves, showing the coordinates of any turning points and where the curve crosses the axes. Include the equations of any asymptotes on your diagrams.

(i) $y = |f(x)|$ (ii) $y = f(|x|)$ (iii) $y = \frac{1}{f(x)}$ (iv) $y^2 = f(x)$

7 $f(x) = \frac{(x+3)}{(x-2)(x+1)}$, $x \neq -1$ and $x \neq 2$

(i) (a) Sketch the curve $y = f(x)$.

(b) Solve the inequality $\frac{(x+3)}{(x-2)(x+1)} < 0$.

(ii) (a) Sketch the curve $y = f(|x|)$.

(b) Solve the inequality $\frac{(|x|+3)}{(|x|-2)(|x|+1)} < 0$.

PS **8** $f(x) = \frac{4ax}{x^2 + a^2}$.

(i) For the curve with equation $y = f(x)$, find

(a) the equation of the asymptote

(b) the range of values that y can take.

(ii) For the curve with equation $y^2 = f(x)$, write down

(a) the equation of the line of symmetry

(b) the maximum and minimum values of y

(c) the set of values of x for which the curve is defined.

9 $f(x) = \frac{9 - 2x}{x + 3}$, $x \neq -3$

(i) Sketch the curve with equation $y = f(x)$.

(ii) State the values of x for which $y = \frac{1}{f(x)}$ is undefined.

Sketch the curve with equation $y = \frac{1}{f(x)}$.

State the coordinates of any points where each curve crosses the axes, and give the equations of any asymptotes.

PS **10** The curve C has equation $y = \dfrac{p(x)}{x+a}$, $x \neq -a$ where $p(x)$ is a polynomial of degree 2 and a is an integer.

The asymptotes of the curve are $x = 1$ and $y = 2x + 5$, and the curve passes through the point (2, 12).

(i) Express the equation of the curve C in the form $y = \dfrac{p(x)}{x+a}$.

(ii) Find the range of values that y can take.

(iii) Sketch the curve with equation $y^2 = \dfrac{p(x)}{x+a}$ where $p(x)$ and a are as found in part (i).

KEY POINTS

1 A rational function is defined as a function that can be expressed in the form $y = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials, and $g(x) \neq 0$.

2 To sketch the graph of $y = f(x)$ follow these steps.

Step 1 Find the intercepts (where the graph cuts the axes).

Step 2 Examine the behaviour of the graph near the vertical asymptotes; these are the lines $x = a$ if $g(a) = 0$ and $f(a) \neq 0$.

Step 3 Examine the behaviour as $x \to \pm\infty$.

Step 4 Find the coordinates of any stationary points.

Step 5 Sketch the graph.

Remember:

- when the order of $g(x)$ is less than the order of $f(x)$ then $y = 0$ is a horizontal asymptote
- when the order of $g(x)$ equals the order of $f(x)$ then $y = c$, for a constant c, is a horizontal asymptote
- when the order of $g(x)$ is greater than the order of $f(x)$ then there is an oblique asymptote. (Use long division to find it.)

3 You can use the discriminant to find the range of a rational function.

4 The graph of $y = f(x)$ can be used to help you solve inequalities and equations.

5 To sketch the graph of $y = |f(x)|$ given the graph of $y = f(x)$ reflect the part of the curve where $y < 0$ in the x-axis.

6 To sketch the graph of $y = f(|x|)$ given the graph of $y = f(x)$ reflect the part of the curve where $x > 0$ in the y-axis.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

7 To sketch the graph of $y = \frac{1}{f(x)}$ given the graph of $y = f(x)$ remember:

- the curves $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect at $y = 1$ and $y = -1$
- $f(x)$ and $\frac{1}{f(x)}$ have the same sign
- any roots of $y = f(x)$ become asymptotes of $y = \frac{1}{f(x)}$
- as $f(x) \to \pm\infty$ then $\frac{1}{f(x)} \to 0$
- there are no x intercepts (roots) for $y = \frac{1}{f(x)}$; any apparent 'roots' (from vertical asymptotes of $y = f(x)$) are discontinuities and should be shown by a small open circle
- when $f(x)$ is increasing then $\frac{1}{f(x)}$ is decreasing
- when $f(x)$ is decreasing then $\frac{1}{f(x)}$ is increasing
- where $f(x)$ has a maximum then $\frac{1}{f(x)}$ is a minimum
- where $f(x)$ has a minimum then $\frac{1}{f(x)}$ has a maximum.

8 When using the graph of $y = f(x)$ to sketch the graph of $y^2 = f(x)$ remember:

- $y^2 = f(x)$ can be thought of as two curves $y = \pm\sqrt{f(x)}$; sketch $y = \sqrt{f(x)}$ first and reflect the result in the x-axis to complete the sketch
- $y^2 = f(x)$ has a line of symmetry in the x-axis
- all y values above the x-axis are replaced by their positive square roots
  - if $y > 1$, then the points get 'pulled down' towards the x-axis ($\sqrt{4} = 2$)
  - if $0 < y < 1$, then the points get 'pulled up' from the x-axis ($\sqrt{\frac{1}{4}} = \frac{1}{2}$)
  - if $y = 1$ the point is invariant (stays the same) ($\sqrt{1} = 1$)
  - at a root (when $y = 0$) the point is invariant (stays the same) and the new curve passes vertically through this point
- any vertical asymptotes stay the same
- any horizontal asymptotes, $x = k$, above the x-axis become the horizontal asymptotes $x = \sqrt{k}$
- any turning point, e.g. (2, 5), above the x-axis remain the same type (i.e. maxima remain as maxima) and the square root of the y coordinate is taken, e.g. $(2, \sqrt{5})$.

LEARNING OUTCOMES

Now that you have finished this chapter, you should be able to

- sketch the graph of a rational function, where the numerator and the denominator are polynomials of degree two or less, including finding
  - equations of vertical and horizontal asymptotes
  - the equation of an oblique asymptote (should it exist)
  - intersections with the x- and y-axes
  - turning points
  - the set of values taken by the function (range)
- use differentiation or the discriminant to determine the range of a function
- use the graph of $y = f(x)$ to sketch the graph of
  - $y = |f(x)|$
  - $y = f(|x|)$
  - $y = \dfrac{1}{f(x)}$
  - $y^2 = f(x)$
- use sketch graphs to help you solve inequalities and equations.