4 Indices and surds

An estate had seven houses;
Each house had seven cats;
Each cat ate seven mice;
Each mouse ate seven grains of wheat.
Wheat grains, mice, cats and houses,
How many were there on the estate?

Ancient Egyptian problem

Discussion point

How can you write down the answer to this problem without doing any calculations?

Indices

The word 'index' (plural indices) has many meanings in real life including a list of names, the index for a book and a price index, but the focus in this chapter is, of course, related to numbers.

Index notation is a shorthand way of writing numbers.

For example, $2 \times 2 \times 2 \times 2 \times 2$ can be written as 2^5.

2 is the base number. 5 is the index or power.

Operations using indices

There are a number of rules that you need to learn when you are working with indices.

Rule: To multiply numbers in index form where the base number is the same, add the indices.
$a^m \times a^n = a^{m+n}$

Multiplying

$$5^4 \times 5^3 = (5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5)$$
$$= 5^7$$

Rule: To divide numbers in index form where the base number is the same, subtract the second index from the first.
$a^m \div a^n = a^{m-n}$

Dividing

$$3^7 \div 3^4 = \frac{3 \times 3 \times 3 \times \cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}$$
$$= 3^3$$

A power raised to a power

Rule: To raise a power to a power, multiply the indices. $(a^m)^n = a^{m\times n}$

$$(2^3)^4 = (2\times2\times2)\times(2\times2\times2)\times(2\times2\times2)\times(2\times2\times2)$$
$$= 2^{12}$$

Index zero

Rule: Any number with an index zero equals one. $a^0 = 1$

Using the rule for division, $6^3 \div 6^3 = 6^{3-3}$

$= 6^0$

However, dividing a number by itself always gives the result 1, so $6^0 = 1$.

Negative indices

$$4^2 \div 4^5 = \frac{4\times4}{4\times4\times4\times4\times4}$$
$$= \frac{1}{4^3}$$

Rule: $\frac{1}{a^m} = a^{-m}$

Using the rule for division, $4^2 \div 4^5 = 4^{2-5}$

$= 4^{-3}$

Worked example

Write $\left(\frac{3}{2}\right)^{-3}$ as a fraction.

Solution

$$\left(\frac{3}{2}\right)^{-3} = 1 \div \left(\frac{3}{2}\right)^3$$
$$= 1 \times \left(\frac{2}{3}\right)^3$$
$$= \frac{8}{27}$$

Fractional indices

What number multiplied by itself equals 5?

The answer to this is usually written as $\sqrt{5}$, but it can also be written in index form.

Let $5^p \times 5^p = 5$

Using the rule for multiplication, $5^{2p} = 5^1$ so $p = \frac{1}{2}$

This gives $\sqrt{5} = 5^{\frac{1}{2}}$

Rule: $\sqrt[n]{a} = a^{\frac{1}{n}}$

Similarly, $\sqrt[3]{5} = 5^{\frac{1}{3}}$

These rules can be combined further to give other rules.

Replacing n by $\frac{1}{n}$ in the rule $(a^m)^n = a^{mn}$ gives the result

$$(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$$

This can also be written as $\left(a^{\frac{1}{n}}\right)^m$ or $(\sqrt[n]{a})^m$

Worked example

Calculate $25^{\frac{3}{2}}$

It is usually more straightforward to use the fractional index first since you are then using smaller numbers and are more likely to recognise the values.

Solution

$$25^{\frac{3}{2}} = \left(25^{\frac{1}{2}}\right)^3$$
$$= \left(\sqrt{25}\right)^3$$
$$= 5^3$$
$$= 125$$

Worked example

Simplify the following, leaving your answers in standard form:

a $(5\times10^5)\times(4\times10^2)$ **b** $(8\times10^5)\div(4\times10^2)$

Solution

a $(5\times10^5)\times(4\times10^2) = (5\times4)\times(10^5\times10^2)$
$$= 20\times10^7$$
$$= 2\times10^8$$

b $(8\times10^5)\div(4\times10^2) = (8\div4)\times(10^5\div10^2)$
$$= 2\times10^3$$

Exercise 4.1

Use a calculator to check your results only.

1 Simplify the following, giving your answers in the form x^n:
a $2^3\times2^7$ **b** $5^{-3}\times5^4$ **c** $3^6\div3^3$
d $6^5\div6^{-4}$ **e** $(4^2)^3$ **f** $(5^2)^{-2}$

2 Simplify the following, leaving your answers in standard form:
a $(3\times10^5)\times(2\times10^9)$ **b** $(2\times10^4)\times(3\times10^{-3})$
c $(8\times10^5)\div10^3$ **d** $(9\times10^9)\div(3\times10^{-3})$

3 Rewrite each of the following as a number raised to a positive integer power:
a 3^{-2} **b** 5^{-4} **c** $\left(\frac{2}{3}\right)^{-3}$ **d** $\left(\frac{1}{3}\right)^{-6}$

4 Simplify the following, leaving your answers in standard form:
a $(5\times10^7)\times(3\times10^{-3})$ **b** $(4\times10^{-2})\times(6\times10^4)$
c $(4\times10^7)\div(8\times10^{-2})$ **d** $(3\times10^3)\div(6\times10^6)$

5 Find the value of each of the following, giving your answer as a whole number or fraction:
a $(3^4\times3^{-2})$ **b** $6^{-6}\times6^6$ **c** $5^5\div5^2$ **d** $(2^3)^4$
e $(3^2)^2$ **f** 7^{-2} **g** $\left(\frac{1}{4}\right)^{-3}$ **h** 2^{-5}
i $\left(\frac{3}{4}\right)^{-2}$ **j** $9^{\frac{1}{2}}$ **k** $81^{\frac{1}{4}}$ **l** $16^{\frac{3}{2}}$
m $27^{\frac{2}{3}}$ **n** $256^{-\frac{1}{4}}$ **o** $128^{-\frac{5}{7}}$

6 Rank each set of numbers in order of increasing size:
a $3^5, 4^4, 5^3$ **b** $2^7, 3^5, 4^4$ **c** $2^{-5}, 3^{-4}, 4^{-3}$

7 Find the value of x in each of the following:
a $\dfrac{3^3 \times 3^6}{3^7 \div 3^5} = 3^x$ **b** $\dfrac{(2^2 \times 2^4)^3}{2 \times 2^3} = 2^x$
c $\dfrac{5^4 \times 5^3 \times 5^2}{5^2 \times 5^x} = 5^6$ **d** $\dfrac{(7^x \times 7^3)^2}{7^4 \div 7^2} = 7^3$
e $\left(\dfrac{1}{2}\right)^x = 8$ **f** $4^x = \dfrac{1}{64}$
g $2^x = 0.125$ **h** $4^x = 0.0625$

8 Simplify the following:
a $3a^2 \times 2a^5$ **b** $6x^4y^2 \times 2xy^{-4}$
c $10b^5 \div 2b^2$ **d** $12p^{-4}q^{-3} \div 3p^2q^2$
e $(4m)^3$ **f** $(2s^2t)^6$

9 Find integers x and y such that $2x \times 3y = 6^4$.

Surds

Surds are irrational numbers that cannot be expressed exactly. $\sqrt{2}$, $\sqrt{3}$, $2+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$ are all examples of surds.

Discussion point

Why is $\sqrt{4}$ not a surd?

Although your calculator will simplify expressions containing square roots of numbers, it is often easier to work with surds in their exact form. You need to know the rules for manipulating surds so you can work with them in an algebraic setting such as $(\sqrt{a}+\sqrt{b})(3\sqrt{a}-4\sqrt{b})$.

Operations using surds

Simplifying surds

To simplify a surd, start by writing the number under the square root sign as a product of two factors, one of which is the largest possible perfect square.

Worked example

Simplify $\sqrt{18}$.

Solution

Writing $\sqrt{18} = \sqrt{6} \times \sqrt{3}$ does not help since neither 6 nor 3 is a perfect square.

$$\begin{aligned}\sqrt{18} &= \sqrt{9 \times 2}\\ &= \sqrt{9} \times \sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

Worked example

Simplify $\sqrt{\frac{2}{9}}$.

Solution

$$\sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{\sqrt{9}}$$
$$= \frac{\sqrt{2}}{3}$$

Adding and subtracting surds

You can add and subtract surds using the same methods as for other algebraic expressions, keeping the rational numbers and the square roots separate.

For example:

$$\sqrt{3} + \sqrt{3} = 2\sqrt{3}$$
$$3\sqrt{5} + 2\sqrt{5} = 5\sqrt{5}$$
$$\sqrt{48} - \sqrt{12} = 4\sqrt{3} - 2\sqrt{3}$$
$$= 2\sqrt{3}$$

Expanding brackets containing surds

To expand brackets containing surds, use the same methods that you use for other algebraic operations.

Worked example

Simplify $\sqrt{3}\left(\sqrt{3} + 2\right)$.

Solution

$$\sqrt{3}\left(\sqrt{3} + 2\right) = \left(\sqrt{3}\right)^2 + 2\sqrt{3}$$
$$= 3 + 2\sqrt{3}$$

Worked example

Simplify $\sqrt{2}(\sqrt{6} + \sqrt{2})$.

Solution

$$\sqrt{2}\left(\sqrt{6} + \sqrt{2}\right) = \sqrt{12} + \left(\sqrt{2}\right)^2$$
$$= \left(\sqrt{4} \times \sqrt{3}\right) + 2$$
$$= 2\sqrt{3} + 2$$

The same principles apply to expanding a pair of brackets. Use whichever method/s you feel most comfortable with.

Worked example

Simplify $(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{2})$.

Solution

$$\begin{aligned}(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{2}) &= (\sqrt{3})^2+\sqrt{6}+\sqrt{6}+(\sqrt{2})^2\\ &= 3+2\sqrt{6}+2\\ &= 5+2\sqrt{6}\end{aligned}$$

When the brackets have the form of the difference of two squares, the result is a numerical value.

Worked example

Simplify $(3+\sqrt{5})(3-\sqrt{5})$.

Solution

$$\begin{aligned}(3+\sqrt{5})(3-\sqrt{5}) &= 3^2-(\sqrt{5})^2\\ &= 9-5\\ &= 4\end{aligned}$$

Rationalising the denominator

Multiplying both the top line (numerator) and the bottom line (denominator) of any fraction by the same amount doesn't change the value of the fraction. You can use this to rationalise the denominator of a fraction involving a surd, i.e. eliminate the surd so that the denominator has an integer value.

Worked example

Simplify $\frac{9}{\sqrt{3}}$.

Solution

Multiplying the numerator and denominator by $\sqrt{3}$

For all values of a, $\sqrt{a}\times\sqrt{a}=a$.

$$\begin{aligned}\frac{9}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} &= \frac{9\sqrt{3}}{3}\\ &= 3\sqrt{3}\end{aligned}$$

Worked example

Simplify $\frac{1}{(\sqrt{5}-1)}$.

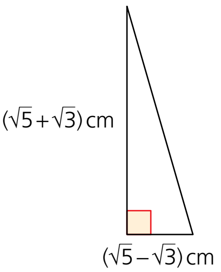
Solution

Using the technique for the difference of two squares, multiply the top and bottom of the fraction by $(\sqrt{5}+1)$.

$$\begin{aligned}\frac{1}{(\sqrt{5}-1)} &= \frac{1}{(\sqrt{5}-1)} \times \frac{(\sqrt{5}+1)}{(\sqrt{5}+1)}\\ &= \frac{\sqrt{5}+1}{5-1}\\ &= \frac{\sqrt{5}+1}{4}\\ &= \frac{1+\sqrt{5}}{4}\end{aligned}$$

Worked example

A right-angled triangle has shorter sides of lengths $(\sqrt{5}+\sqrt{3})$ cm and $(\sqrt{5}-\sqrt{3})$ cm. Work out the length of the hypotenuse.



Solution

Let the length of the hypotenuse be h.

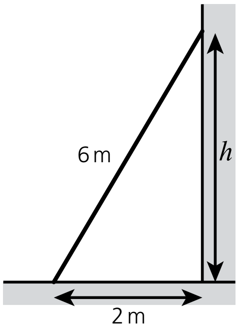
Using Pythagoras' theorem

$$\begin{aligned}h^2 &= (\sqrt{5}+\sqrt{3})^2 + (\sqrt{5}-\sqrt{3})^2\\ &= (5+2\sqrt{15}+3)+(5-2\sqrt{15}+3)\\ &= 8+2\sqrt{15}+8-2\sqrt{15}\\ &= 16\end{aligned}$$

The length of the hypotenuse is 4 cm.

Worked example

A ladder of length 6 m is placed 2 m from a vertical wall at the side of a house. How far up the wall does the ladder reach? Give your answer as a surd in its simplest form.



Solution

Using Pythagoras' theorem

$$6^2 = 2^2 + h^2$$

$$\Rightarrow \quad 36 = 4 + h^2$$

$$\Rightarrow \quad 32 = h^2$$

h must be positive.

$$\Rightarrow \quad h = \sqrt{32}$$

$$= 4\sqrt{2}$$

The ladder reaches $4\sqrt{2}$ metres up the wall.

Exercise 4.2

1 Write each of the following in its simplest form:

a $\sqrt{12}$ b $\sqrt{75}$ c $\sqrt{300}$

d $3\sqrt{5} + 6\sqrt{5}$ e $\sqrt{48} + \sqrt{27}$ f $3\sqrt{45} - 2\sqrt{20}$

2 Express each of the following as the square root of a single number:

a $3\sqrt{6}$ b $5\sqrt{5}$ c $12\sqrt{3}$ d $10\sqrt{17}$

3 Simplify the following:

a $\sqrt{\frac{25}{49}}$ b $\sqrt{\frac{24}{9}}$ c $\sqrt{\frac{12}{15}}$ d $\sqrt{\frac{6}{121}}$

4 Simplify the following by collecting like terms:

a $\left(3+\sqrt{2}\right)+\left(5+4\sqrt{2}\right)$ b $4\left(\sqrt{3}-1\right)+4\left(\sqrt{3}+1\right)$

5 Expand and simplify:

a $\left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right)$ b $\sqrt{3}\left(5-\sqrt{3}\right)$

c $\left(4+\sqrt{2}\right)^2$ d $\left(\sqrt{6}-\sqrt{3}\right)\left(\sqrt{6}-\sqrt{3}\right)$

6 Rationalise the denominators, giving each answer in its simplest form:

a $\frac{1}{\sqrt{6}}$ b $\frac{12}{\sqrt{3}}$ c $\frac{\sqrt{6}}{2\sqrt{2}}$

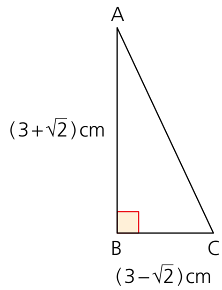
d $\frac{1}{\left(\sqrt{5}-\sqrt{2}\right)}$ e $\frac{3+\sqrt{2}}{4-\sqrt{2}}$ f $\frac{3-\sqrt{5}}{5+\sqrt{5}}$

7 Write the following in the form $a + b\sqrt{c}$ where c is an integer and a and b are rational numbers:

a $\frac{1+\sqrt{2}}{3-\sqrt{2}}$ b $\frac{3\sqrt{5}}{3+\sqrt{5}}$ c $\frac{2\sqrt{6}}{\sqrt{6}-2}$

Exercise 4.2 (cont)

8 Work out the length of AC.



9 A square has sides of length x cm and diagonals of length 12 cm. Use Pythagoras' theorem to find the exact value of x and work out the area of the square.

10 An equilateral triangle has sides of length $\sqrt{3}$ cm.
Work out:
a the height of the triangle
b the area of the triangle in its simplest surd form.

Past-paper questions

1 Without using a calculator, find the positive root of the equation
$(5-2\sqrt{2})x^2-(4+2\sqrt{2})x-2=0$
giving your answer in the form $a+b\sqrt{2}$, where a and b are integers. [6]

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2 (a) Solve the equation $16^{3x-2}=8^{2x}$. [3]

(b) Given that $\dfrac{\sqrt{a^{\frac{4}{3}}b^{-\frac{2}{5}}}}{a^{-\frac{1}{3}}b^{\frac{3}{5}}}=a^pb^q$, find the value of p and of q. [2]

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Paper 11 Q4 June 2011
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Paper 11 Q4 June 2011

3 (i) Given that $2^{5x}\times 4^y=\frac{1}{8}$, show that $5x+2y=-3$. [3]

(ii) Solve the simultaneous equations $2^{5x}\times 4^y=\frac{1}{8}$ and $7^x\times 49^{2y}=1$. [4]

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Learning outcomes

Now you should be able to:

★ perform simple operations with indices and with surds, including rationalising the denominator.

Key points

✔ Use these rules for manipulating indices (powers).

- Multiplication: $a^m \times a^n = a^{m+n}$
- Division: $a^m \div a^n = a^{m-n}$
- Power of a power: $(a^m)^n = a^{mn}$
- Power zero: $a^0 = 1$
- Negative indices: $a^{-m} = \frac{1}{a^m}$
- Fractional indices: $a^{\frac{1}{n}} = \sqrt[n]{a}$

✔ Use these rules for simplifying surds (square roots).

- Leave the smallest number possible under the square root sign, e.g. $\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$
- Expand a surd expression in brackets in the same way any other algebraic expression.
- To rationalise the denominator of a surd:

  (i) If the denominator contains a single term, multiply numerator and denominator by that term,
  e.g. $\frac{7}{\sqrt{5}} = \frac{7 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{7\sqrt{5}}{5}$

  (ii) If the denominator contains two terms, multiply numerator and denominator by a similar expression with the opposite sign, e.g. $\frac{7}{6-\sqrt{2}} = \frac{7}{(6-\sqrt{2})} \times \frac{(6+\sqrt{2})}{(6+\sqrt{2})} = \frac{7(6+\sqrt{2})}{6^2-(\sqrt{2})^2} = \frac{7(6+\sqrt{2})}{32}$