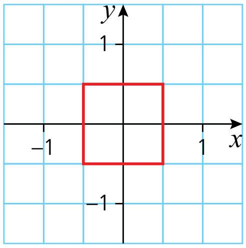
3 Equations, inequalities and graphs

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that that we ignore its true merit.

Pierre-Simon, Marquis de Laplace (1749 – 1827)

The picture shows a quadrat. This is a tool used by biologists to select a random sample of ground; once it is in place they will make a record of all the plants and creatures living there. Then they will throw the quadrat so that it lands somewhere else.



This diagram illustrates a 1 metre square quadrat. The centre point is taken to be the origin and the sides to be parallel to the x- and y-axes.

Discussion point

What is the easiest way to describe the region it covers?

Many practical situations involve the use of inequalities.

How economical is a Formula 1 car?

The Monaco Grand Prix, consisting of 78 laps and a total distance of approximately 260 km, is a well-known Formula 1 race. In 2017 it was won by Sebastian Vettel in 1 hour 44 minutes. Restrictions on the amount and use of fuel mean that drivers need to manage the performance of their car very carefully throughout the race.

A restriction in 2017 was that the total amount of fuel used during the race was limited to 105kg which is approximately 140 litres.

Using: f to denote the total amount of fuel used in litres

d to represent the distance travelled in kilometres

E to represent the fuel economy in litres per kilometre $\left(E = \frac{f}{d}\right)$

the restriction can be represented as $E \leqslant \frac{140}{260}$

$$\Rightarrow E \leqslant 0.538$$

This shows that, at worst, the fuel economy of Vettel's Ferrari Formula 1 car is 0.538 litres per kilometre.

Discussion point

How does this compare with an average road car?

Modulus functions and graphs

For any real number x, the modulus of x is denoted by $|x|$ and is defined as:
$|x| = x$ if $x \geqslant 0$
$|x| = -x$ if $x < 0$.

For any real number, the **absolute value**, or **modulus**, is its positive size whether that number is positive or negative. It is denoted by a vertical line on each side of the quantity. For example, $|5| = 5$ and $|-5| = 5$ also. The absolute value of a number will always be positive or zero. It can be thought of as the distance between that point on the x-axis and the origin.

You have already met graphs of the form $y = 3x + 4$ and $y = x^2 - 3x - 4$. However you might not be as familiar with graphs of the form $y = |3x + 4|$ and $y = |x^2 - 3x - 4|$.

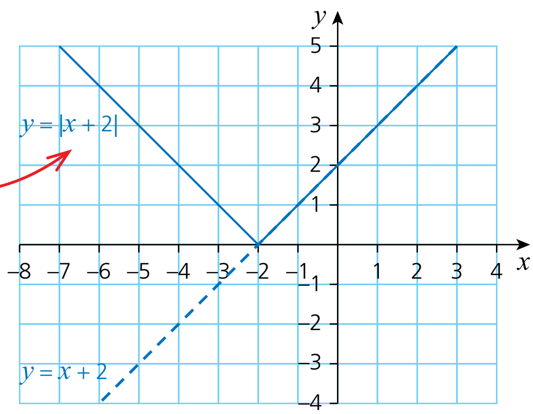
Worked example

Set up a table for the graphs $y = x + 2$ and $y = |x + 2|$ for $-6 \leqslant x \leqslant 2$. Draw both graphs on the same axes.

Solution

y	-6	-5	-4	-3	-2	-1	0	1	2
$x + 2$	-4	-3	-2	-1	0	1	2	3	4
$\lvert x + 2\rvert$	4	3	2	1	0	1	2	3	4

The equation of this part of the graph is $y = -(x + 2)$.



Notice that effect of taking the modulus is a transformation in which the positive part of the original graph (above the x-axis) remains the same and the negative part of the original graph (below the x-axis) is reflected in the x-axis.

Solving modulus equations

Worked example

Solve the equation $|2x + 3| = 5$

a graphically

b algebraically.

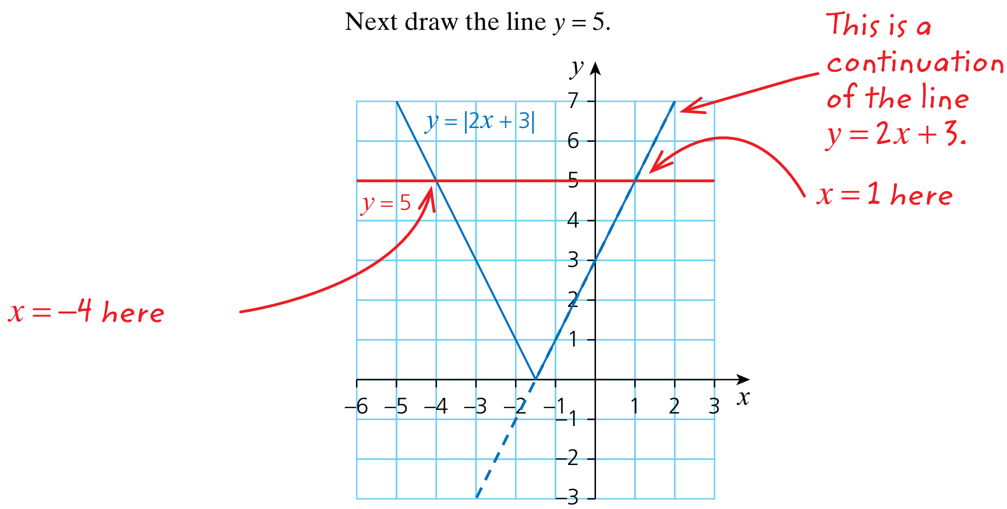
Solution

a First draw the graph of $y = 2x + 3$.

Start by choosing three values of x and calculating the corresponding values of y, for example, $(-2, -1)$, $(0, 3)$ and $(2, 7)$.

Then reflect in the x-axis any part of the graph that is below the x-axis to give the graph of $y = |2x + 3|$.

Next draw the line $y = 5$.



The solution is given by the values of x where the V-shaped graph meets the line $y = 5$

$\Rightarrow x = 1$ or $x = -4$.

b $|2x + 3| = 5 \Rightarrow 2x + 3 = 5$ or $2x + 3 = -5$

$\Rightarrow 2x = 2$ or $2x = -8$

$\Rightarrow x = 1$ or $x = -4$

Discussion point

Notice that in the solution three points are used to draw the straight line when only two are necessary. Why is this good practice?

Either of these methods can be extended to find the points where two V-shaped graphs intersect. However, the graphical method will not always give an accurate solution.

Worked example

Solve the equation $|2x + 5| = |x - 4|$.

Solution

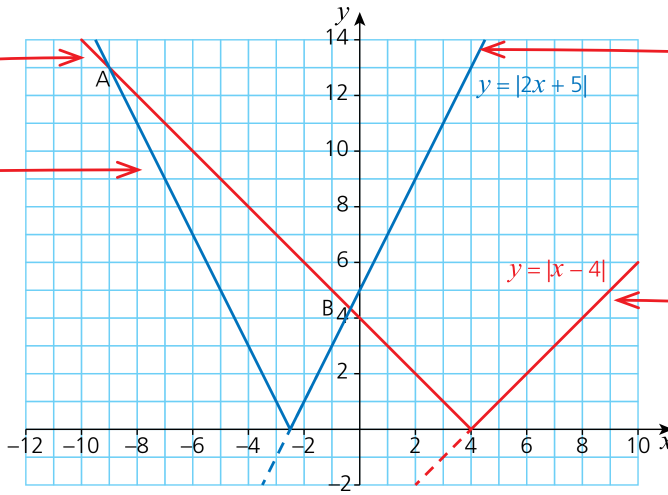
Start by drawing the graphs of $y = |2x + 5|$ and $y = |x - 4|$ on the same axes.

This is part of the line $y = -(x - 4)$.

This is part of the line $y = -(2x + 5)$.

This is part of the line $y = 2x + 5$.

This is part of the line $y = x - 4$.



This shows a failing of the graphical method. However, the graph is useful in determining the equation of the line required for an algebraic solution.

The graph shows that the point A is (–9, 13), but the coordinates of B are not clear.

The graph shows that both points of intersection occur where the reflected part of the line $y = x - 4$, i.e. the line $y = -(x - 4)$ intersects the graph of $y = |2x + 5|$.

At A, $y = 4 - x$ meets $y = -2x - 5$

$\Rightarrow 4 - x = -2x - 5$

$\Rightarrow 2x - x = -5 - 4$

$\Rightarrow x = -9$

When $x = -9$, $y = 4 - (-9) = 13$, i.e. A is the point (–9, 13).

At B, $y = 4 - x$ meets $y = 2x + 5$

$\Rightarrow 2x + 5 = 4 - x$

$\Rightarrow 3x = -1$

$\Rightarrow x = -\frac{1}{3}$

When $x = -\frac{1}{3}$, $y = 4 - \left(-\frac{1}{3}\right) = 4\frac{1}{3}$, i.e. B is the point $\left(-\frac{1}{3}, 4\frac{1}{3}\right)$.

Exercise 3.1

For questions 1–3, sketch each pair of graphs on the same axes.

1 **a** $y = x$ and $y = |x|$ **b** $y = x - 1$ and $y = |x - 1|$
c $y = x - 2$ and $y = |x - 2|$

2 **a** $y = 2x$ and $y = |2x|$ **b** $y = 2x - 1$ and $y = |2x - 1|$
c $y = 2x - 2$ and $y = |2x - 2|$

3 **a** $y = 2 - x$ and $y = |2 - x|$ **b** $y = 3 - x$ and $y = |3 - x|$
c $y = 4 - x$ and $y = |4 - x|$

Exercise 3.1 (cont)

4 **a** Draw the graph of $y = |x + 1|$.
b Use the graph to solve the equation $|x + 1| = 5$.
c Use algebra to verify your answer to part **b**.

5 **a** Draw the graph of $y = |x - 1|$.
b Use the graph to solve the equation $|x - 1| = 5$.
c Use algebra to verify your answer to part **b**.

6 **a** Draw the graph of $y = |2x + 3|$.
b Use the graph to solve the equation $|2x + 3| = 7$.
c Use algebra to verify your answer to part **b**.

7 **a** Draw the graph of $y = |2x - 3|$.
b Use the graph to solve the equation $|2x - 3| = 7$.
c Use algebra to verify your answer to part **b**.

8 Solve the equation $|x + 1| = |x - 1|$ both graphically and algebraically.

9 Solve the equation $|x + 5| = |x - 5|$ both graphically and algebraically.

10 Solve the equation $|2x + 4| = |2x - 4|$ both graphically and algebraically.

Solving modulus inequalities

When illustrating an inequality in one variable:

- An open circle at the end of a line shows that the end point is excluded.
- A solid circle at the end of a line shows that the end point is included.
- The line is drawn either in colour or as a solid line.

For example, the inequality $-2 < x \leqslant 3$ is shown as:

Worked example

a Solve algebraically the inequality $|x - 3| > 2$.

b Illustrate the solution on a number line.

An open circle is used to show that the value there is not part of the solution.

Solution

a $|x - 3| > 2 \Rightarrow x - 3 > 2$ or $x - 3 < -2$

$\Rightarrow x > 5$ or $x < 1$

The blue lines show the required parts of the number line.

b

Worked example

Write the inequality $-3 \leqslant x \leqslant 9$ in the form $|x - a| \leqslant b$ and show a and b on a number line.

Solution

$|x - a| \leqslant b \quad \Rightarrow \quad -b \leqslant x - a \leqslant b$

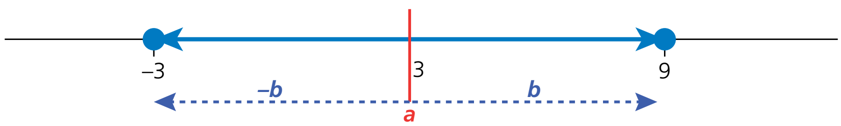
$\quad \Rightarrow \quad a - b \leqslant x \leqslant a + b$

You are finding the values of x **within** $a \pm b$.

Solve $a + b = 9$ and $a - b = -3$ simultaneously.

Adding: $2a = 6$, so $a = 3$

Subtracting: $2b = 12$, so $b = 6$



Substituting in $a - b \leqslant x \leqslant a + b$ gives $-3 \leqslant x \leqslant 9$.
Substituting in $|x - a| \leqslant b$ gives $|x - 3| \leqslant 6$.

Worked example

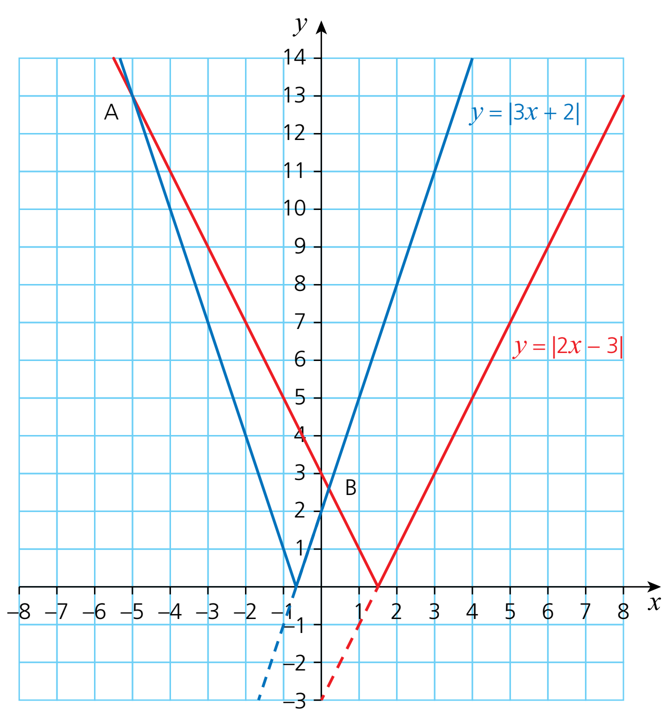
Solve the inequality $|3x + 2| \leqslant |2x - 3|$.

Solution

Draw the graphs of $y = |3x + 2|$ and $y = |2x - 3|$. The inequality is true for values of x where the unbroken blue line is below or crosses the unbroken red line, i.e. between (and including) the points A and B.

Draw the line $y = 3x + 2$ as a straight line through $(0, 2)$ with a gradient of $+3$. Reflect in the x-axis the part of the line that is below this axis.

Draw the line $y = 2x - 3$ as a straight line through $(0, -3)$ with a gradient of $+2$. Reflect in the x-axis the part of the line that is below this axis.



The graph shows that $x = -5$ at A, but the exact value for x at B is not clear. The algebraic solution gives a more precise value.

At A, $-(3x + 2) = -(2x - 3) \Rightarrow 3x + 2 = 2x - 3$

$$\Rightarrow x = -5$$

Substituting in either of the equations gives $y = 13$, so A is the point $(-5, 13)$.

At B, $3x + 2 = -(2x - 3) \Rightarrow 3x + 2 = -2x + 3$

$$\Rightarrow 5x = 1$$

$$\Rightarrow x = 0.2$$

Substituting in either equation gives $y = 2.6$, so B is the point $(0.2, 2.6)$.

The inequality is satisfied for values of x between A and B, i.e. for $-5 \leqslant x \leqslant 0.2$.

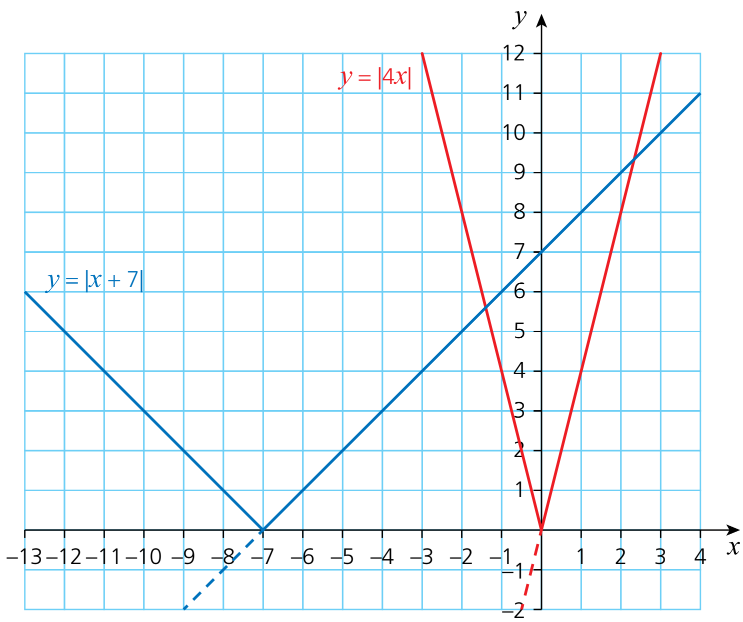
Worked example

Solve the inequality $|x + 7| < |4x|$.

Solution

The question does not stipulate a particular method, so start with a sketch graph.

The sketch graph shows that the points where the two graphs intersect do not have integer coordinates. This means that a graphical method is unlikely to give an accurate solution.



Use algebra to find the points when $|x + 7| = |4x|$, i.e. when $x + 7 = 4x$ and when $x + 7 = -4x$.

Discussion point

Why is it sufficient to consider only these two cases? Why do you not need to consider when $-(x + 7) = 4x$?

$x + 7 = 4x$ when $x = \frac{7}{3}$.

$x + 7 = -4x$ when $x = -\frac{7}{5}$.

Think about a point to the left of $x = -\frac{7}{5}$, such as $x = -2$.

This tells you that part of the solution is to the left of $x = -\frac{7}{5}$, i.e. $x < -\frac{7}{5}$.

When $x = -2$, $|x + 7| < |4x|$ gives $5 < 8$. This is true so the inequality is satisfied.

Next think about a value of x in the interval $\left(-\frac{7}{5}, \frac{7}{3}\right)$, for example, $x = 0$.

Any other value in this interval will also give a false result.

When $x = 0$, $|x + 7| < |4x|$ gives $7 < 0$, which is false.

Finally consider a value greater than $\frac{7}{3}$, for example, $x = 3$.

When $x = 3$, $|x + 7| < |4x|$ gives $10 < 12$. This is true so the inequality is satisfied.

Therefore, the solution is $x < -\frac{7}{5}$ or $x > \frac{7}{3}$.

Inequalities in two dimensions are illustrated by regions. For example, $x > 2$ is shown by the part of the x–y plane to the right of the line $x = 2$ and $x < -1$ by the part to the left of the line $x = -1$.

If you are asked to illustrate the region $x \geqslant 2$, then the line $x = 2$ must be included as well as the region $x > 2$.

Note

- When the boundary line is included, it is drawn as a **solid line**; when it is excluded, it is drawn as a **dotted line**.
- The answer to an inequality of this type is **a region of the x–y plane**, not simply a set of points. It is common practice to specify the region that you want (called the **feasible region**) by **shading out the unwanted region**. This keeps the feasible region clear so that you can see clearly what you are working with.

Worked example

Illustrate the inequality $3y - 2x \geqslant 0$ on a graph.

Solution

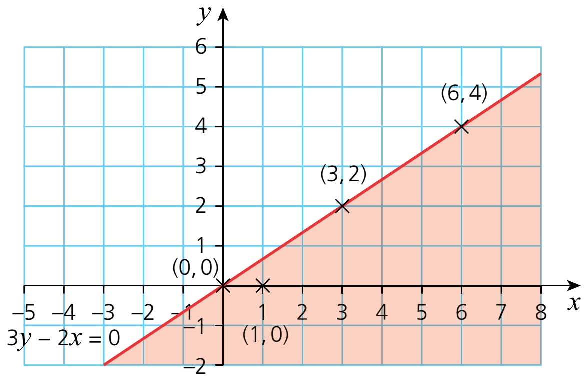
Draw the line $3y - 2x = 0$ as a **solid** line through (0, 0), (3, 2) and (6, 4).

Discussion point

Why are these points more suitable than, for example, $\left(1, \frac{2}{3}\right)$?

Choose a point which is not on the line as a test point, for example, (1, 0).

Using these values, $3y - 2x = -2$. This is clearly not true, so this point is not in the feasible region. Therefore shade out the region containing the point (1, 0).



Exercise 3.2

1 Write each of the following inequalities in the form $|x - a| \leqslant b$:

a $-3 \leqslant x \leqslant 15$ **b** $-4 \leqslant x \leqslant 16$ **c** $-5 \leqslant x \leqslant 17$

2 Write each of the following expressions in the form $a \leqslant x \leqslant b$:

a $|x - 1| \leqslant 2$ **b** $|x - 2| \leqslant 3$ **c** $|x - 3| \leqslant 4$

3 Solve the following inequalities and illustrate each solution on a number line:

a $|x - 1| < 4$ **b** $|x - 1| > 4$ **c** $|2x + 3| < 5$
d $|2x + 3| > 5$

4 Illustrate each of the following inequalities graphically by shading the unwanted region:

a $y - 2x > 0$ **b** $y - 2x \leqslant 0$ **c** $2y - 3x > 0$
d $2y - 3x \leqslant 0$

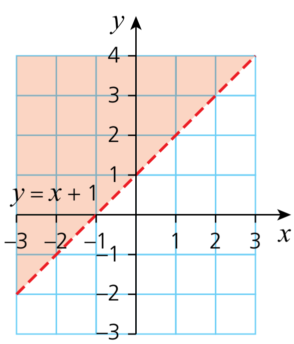
5 Solve the following inequalities:

i graphically **ii** algebraically.

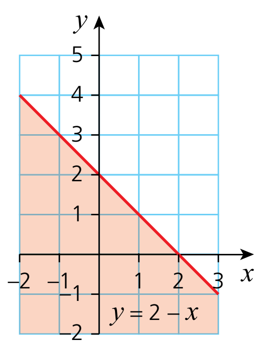
a $|x - 1| < |x + 1|$ **b** $|x - 1| > |x + 1|$
c $|2x - 1| \leqslant |2x + 1|$ **d** $|2x - 1| \geqslant |2x + 1|$

6 Each of the following graphs represents an inequality. Name the inequality.

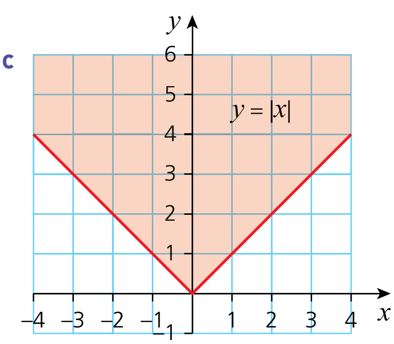
a



b



c



Using substitution to solve quadratic equations

Sometimes you will meet an equation which includes a square root. Although this is not initially a quadratic equation, you can use a substitution to solve it in this way, as shown in the following example.

Worked example

Use the substitution $x = u^2$ to solve the equation $x - 3\sqrt{x} = -2$.

Solution

Substituting $x = u^2$ in the equation $x - 3\sqrt{x} = -2$ gives $u^2 - 3u = -2$

$\Rightarrow u^2 - 3u + 2 = 0$

Factorising → $\Rightarrow (u - 1)(u - 2) = 0$

$\Rightarrow u = 1$ or $u = 2$

Since $x = u^2$, $x = 1$ or $x = 4$.

Checking these values → When $x = 1$, $1 - 3\sqrt{1} = -2$, so $x = 1$ is a valid solution.

When $x = 4$, $4 - 3\sqrt{4} = -2$, so $x = 4$ is also a valid solution.

It is always advisable to check possible solutions, since in some cases not all values of u will give a valid solution to the equation, as shown in the following example.

Worked example

Solve the equation $x - \sqrt{x} = 6$.

Solution

Substituting $x = u^2$ in the equation $x - \sqrt{x} = 6$ gives $u^2 - u = 6$

$\Rightarrow u^2 - u - 6 = 0$

Factorising → $\Rightarrow (u - 3)(u + 2) = 0$

$\Rightarrow u = 3$ or $u = -2$

Since $x = u^2$, $x = 9$ or $x = 4$.

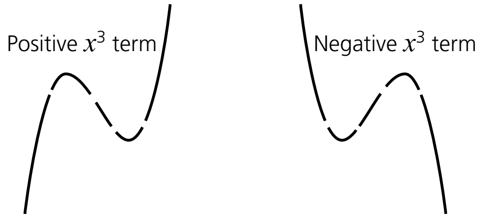
Checking these values → When $x = 9$, $9 - \sqrt{9} = 6$, so $x = 9$ is a possible solution.

When $x = 4$, $4 - \sqrt{4} = 2$, so reject $x = 4$ as a possible solution.

The only solution to this equation is $x = 9$.

Using graphs to solve cubic inequalities

Cubic graphs have distinctive shapes determined by the coefficient of x^3.



The centre part of each of these curves may not have two distinct turning points like those shown above, but may instead 'flatten out' to give a **point of inflection.** When the modulus of a cubic function is required, any part of the curve below the x-axis is reflected in that axis.

Worked example

You are asked for a sketch graph, so although it must show the main features, it does not need to be absolutely accurate. You may find it easier to draw the curve first, with the positive x^3 term determining the shape of the curve, and then position the x-axis so that the distance between the first and second intersections is about half that between the second and third, since these are 3 and 6 units respectively.

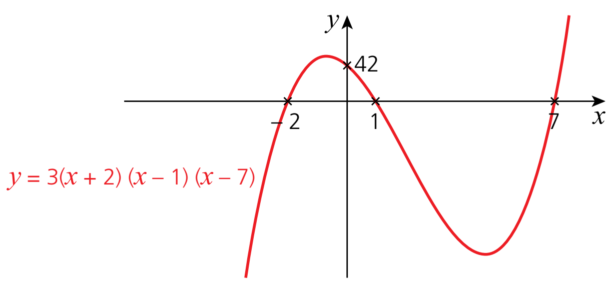
a Sketch the graph of $y = 3(x+2)(x-1)(x-7)$. Identify the points where the curve cuts the axes.

b Sketch the graph of $y = |3(x+2)(x-1)(x-7)|$.

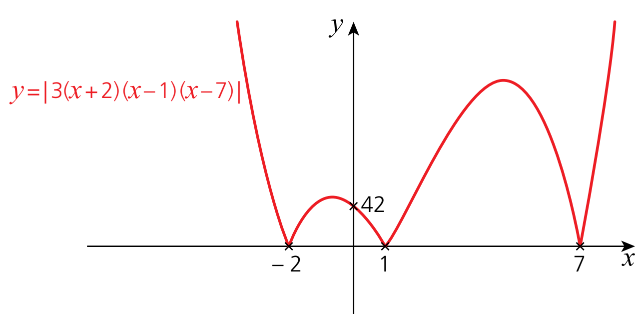
Solution

a The curve crosses the x-axis at –2, 1 and 7. Notice that the distance between consecutive points is 3 and 6 units respectively, so the y-axis is between the points –2 and 1 on the x-axis, but closer to the 1.

The curve crosses the y-axis when $x = 0$, i.e. when $y = 3(2)(-1)(-7) = 42$.



b To obtain a sketch of the modulus curve, reflect any part of the curve which is below the x-axis in the x-axis.



Worked example

Solve the inequality $3(x+2)(x-1)(x-7) \leqslant -100$ graphically.

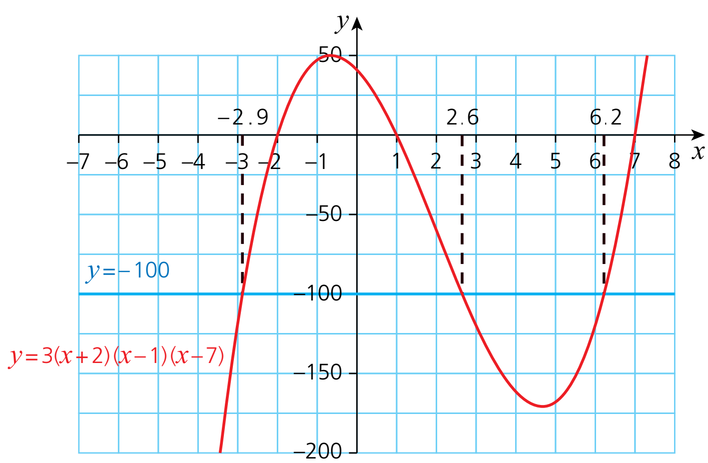
Solution

Because you are solving the inequality graphically, you will need to draw the curve as accurately as possible on graph paper, so start by drawing up a table of values.

$y = 3(x+2)(x-1)(x-7)$

x	–3	–2	–1	0	1	2	3	4	5	6	7	8
$(x+2)$	–1	0	1	2	3	4	5	6	7	8	9	10
$(x-1)$	–4	–3	–2	–1	0	1	2	3	4	5	6	7
$(x-7)$	–10	–9	–8	–7	–6	–5	–4	–3	–2	–1	0	1
x	–120	0	48	42	0	–60	–120	–162	–168	–120	0	210

The solution is given by the values of x that correspond to the parts of the curve on or below the line $y = -100$.



From the graph, the solution is $x \leqslant -2.9$ or $2.6 \leqslant x \leqslant 6.2$.

Exercise 3.3

1 Where possible, use the substitution $x = u^2$ to solve the following equations:

a $x - 4\sqrt{x} = -4$ **b** $x + 2\sqrt{x} = 8$

c $x - 2\sqrt{x} = 15$ **d** $x + 6\sqrt{x} = -5$

2 Sketch the following graphs, indicating the points where they cross the x-axis:

a $y = x(x-2)(x+2)$ **b** $y = |x(x-2)(x+2)|$

c $y = 3(2x-1)(x+1)(x+3)$ **d** $y = |3(2x-1)(x+1)(x+3)|$

3 Solve the following equations graphically. You will need to use graph paper.

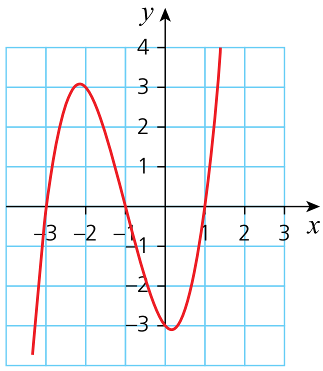
a $x(x+2)(x-3) \geqslant 1$ **b** $x(x+2)(x-3) \leqslant -1$

c $(x+2)(x-1)(x-3) > 2$ **d** $(x+2)(x-1)(x-3) < -2$

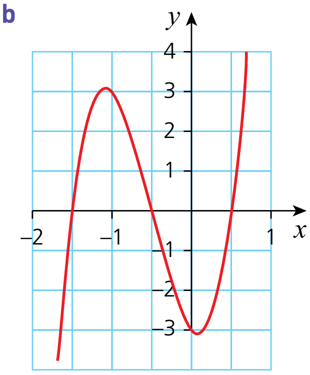
Exercise 3.3 (cont)

4 Identify the following cubic graphs:

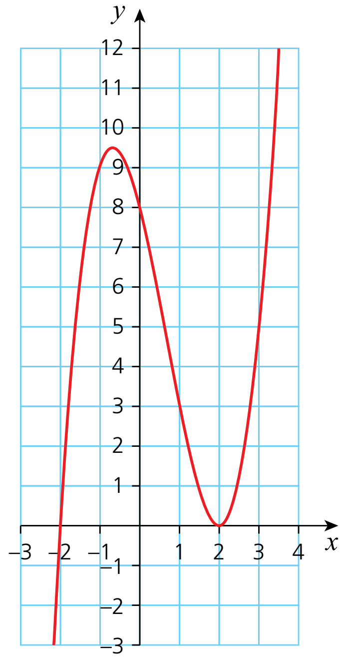
a



b

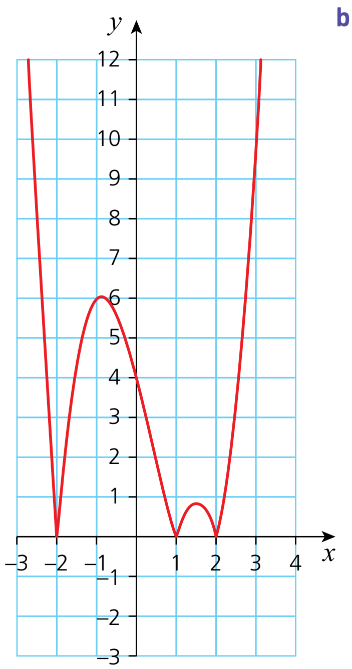


c

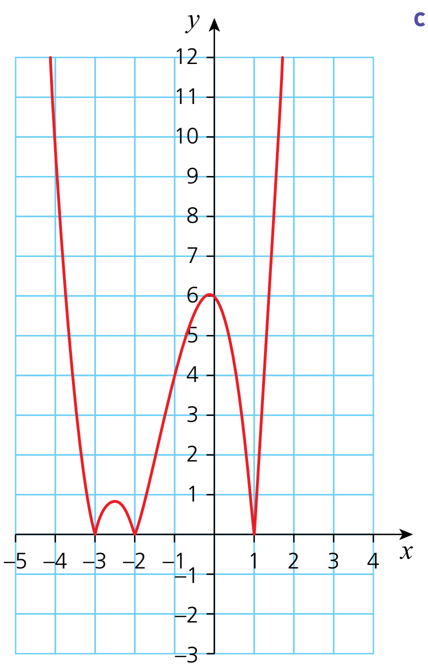


5 Identify these graphs. (They are the moduli of cubic graphs.)

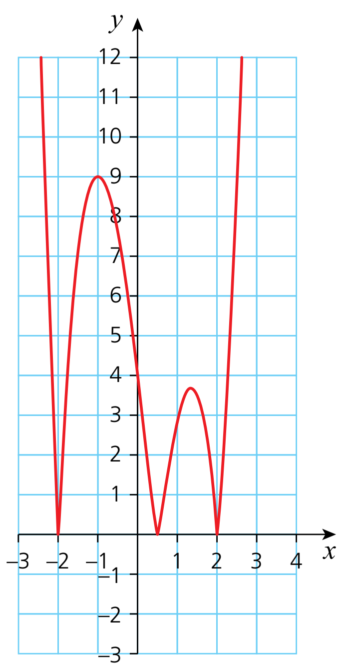
a



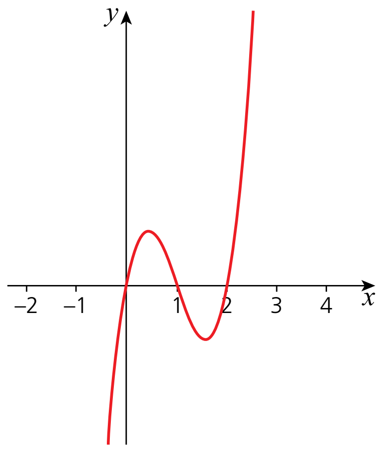
b



c



6 Why is it not possible to identify the following graph without further information?



Past-paper questions

1 **(i)** Sketch the graph of $y = |(2x + 3)(2x - 7)|$. [4]

(ii) How many values of x satisfy the equation $|(2x + 3)(2x - 7)| = 2x$? [2]

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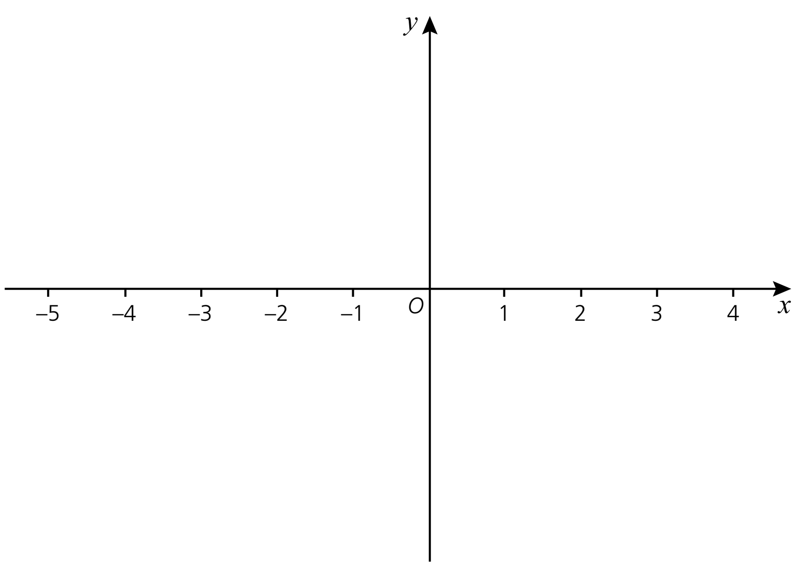
Cambridge IGCSE Additional Mathematics 0606
Paper 23 Q6 November 2011

2 **(i)** On a grid like the one below, sketch the graph of $y = |(x - 2)(x + 3)|$ for $-5 \leqslant x \leqslant 4$, and state the coordinates of the points where the curve meets the coordinate axes. [4]



(ii) Find the coordinates of the stationary point on the curve $y = |(x - 2)(x + 3)|$. [2]

(iii) Given that k is a positive constant, state the set of values of k for which $|(x - 2)(x + 3)| = k$ has 2 solutions only. [1]

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3 Solve the inequality $9x^2 + 2x - 1 < (x + 1)^2$. [3]

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Learning outcomes

Now you should be able to:

★ solve graphically or algebraically equations of the type $|ax + b| = c$ $(c \geqslant 0)$ and $|ax + b| = |cx + d|$
★ solve graphically or algebraically inequalities of the type $|ax + b| > c$ $(c \geqslant 0)$, $|ax + b| \leqslant c$ $(c > 0)$and $|ax + b| \leqslant (cx + d)$
★ use substitution to form and solve a quadratic equation in order to solve a related equation
★ sketch the graphs of cubic polynomials and their moduli, when given in factorised form $y = k(x - a)(x - b)(x - c)$
★ solve cubic inequalities in the form $k(x-a)(x-b)(x-c) \leqslant d$ graphically.

Key points

✔ For any real number x, the **modulus** of x is denoted by $|x|$ and is defined as:

$$|x| = x \text{ if } x \geqslant 0$$
$$|x| = -x \text{ if } x < 0.$$

✔ A modulus equation of the form $|ax + b| = b$ can be solved either graphically or algebraically.
✔ A modulus equation of the form $|ax + b| = |cx + d|$ can be solved graphically by first drawing both graphs on the same axes and then, if necessary, identifying the solution algebraically.

- ✔ A modulus inequality of the form $|x - a| < b$ is equivalent to the inequality $a - b < x < a + b$ and can be illustrated on a number line with an open circle marking the ends of the interval to show that these points are not included. For $|x - a| \leqslant b$, the interval is the same but the end points are marked with solid circles.
- ✔ A modulus inequality of the form $|x - a| > b$ or $|x - a| \geqslant b$ is represented by the parts of the line outside the intervals above.
- ✔ A modulus inequality in two dimensions is identified as a region on a graph, called the **feasible region.** It is common practice to shade out the region not required to keep the feasible region clear.
- ✔ It is sometimes possible to solve an equation involving both x and $\sqrt{x}$ by making a substitution of the form $x = u^2$. You must check all answers in the original equation.
- ✔ The graph of a cubic function has a distinctive shape determined by the coefficient of x^3.