2 Quadratic functions

One really can't argue with a mathematical theorem.

Stephen Hawking (1942–2018)

Early mathematics focused principally on arithmetic and geometry. However, in the sixteenth century a French mathematician, François Viète, started work on 'new algebra'. He was a lawyer by trade and served as a privy councillor to both Henry III and Henry IV of France. His innovative use of letters and parameters in equations was an important step towards modern algebra.

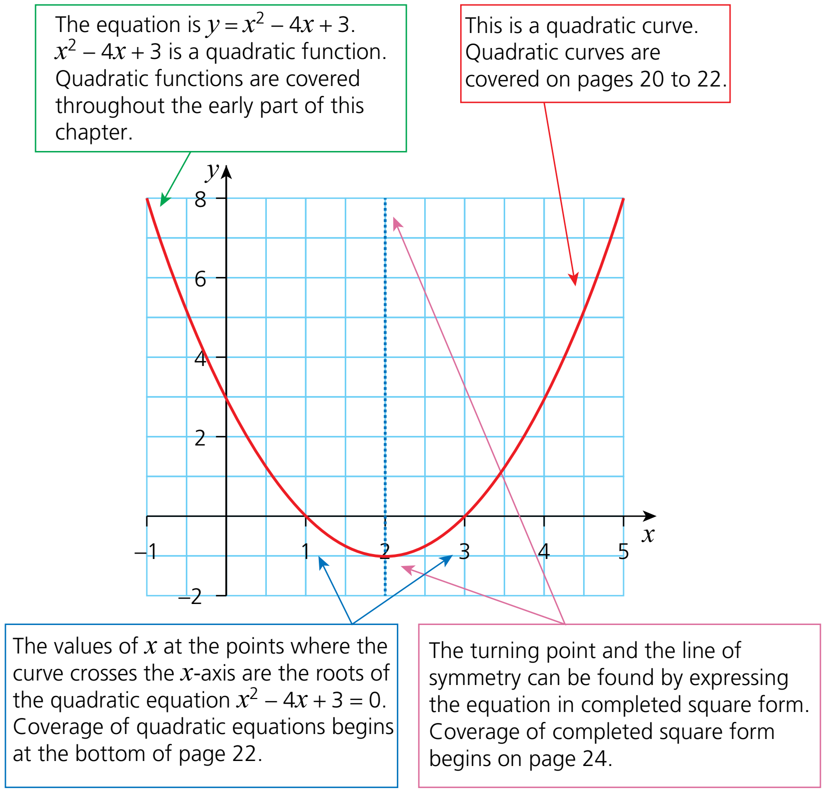
François Viète (1540–1603)

Discussion point

Viète presented methods of solving equations of second, third and fourth degrees and discovered the connection between the positive roots of an equation and the coefficients of different powers of the unknown quantity. Another of Viete's remarkable achievements was to prove that claims that a circle could be squared, an angle trisected and the cube doubled were untrue. He achieved all this, and much more, using only a ruler and compasses, without the use of either tables or a calculator! In order to appreciate the challenges Viète faced, try to solve the quadratic equation $2x^2 - 8x + 5 = 0$ without using a calculator. Give your answers correct to two decimal places.

This chapter is about quadratic functions and covers a number of related themes.

The graph below illustrates these themes:



Maximum and minimum values

A **polynomial** is an expression in which, with the exception of a constant, the terms are positive integer powers of a variable. The highest power is the **order** of the polynomial.

A **quadratic function** or expression is a polynomial of order 2. $x^2 + 3$, a^2 and $2y^2 - 3y + 5$ are all quadratic expressions. Each expression contains only one variable (letter), and the highest power of that variable is 2.

The graph of a quadratic function is either ∪-shaped or ∩-shaped. Think about the expression $x^2 + 3x + 2$. When the value of x is very large and positive, or very large and negative, the x^2 term dominates the expression, resulting in large positive values. Therefore the graph of the function is ∪-shaped.

Similarly, the $-2x^2$ term dominates the expression $5 - 4x - 2x^2$ for both large positive and large negative values of x giving negative values of the expression for both. Therefore the graph of this function is ∩-shaped.

Although many of the quadratic equations that you will meet will have three terms, you will also meet quadratic equations with only two, or even one term. These fall into two main categories.

1 Equations with no constant term, for example, $2x^2 - 5x = 0$.
This has x as a common factor so factorises to $x(2x - 5) = 0$

$$\Rightarrow x = 0 \text{ or } 2x - 5 = 0$$
$$\Rightarrow x = 0 \text{ or } x = 2.5$$

2 Equations with no 'middle' term, which come into two categories:

i The sign of the constant term is negative, for example, $a^2 - 9 = 0$ and $2a^2 - 7 = 0$.

$a^2 - 9 = 0$ factorises to $(a + 3)(a - 3) = 0$
$$\Rightarrow a = -3 \text{ or } a = 3$$
$$2a^2 - 7 = 0 \Rightarrow a^2 = 3.5$$
$$\Rightarrow a = \pm\sqrt{3.5}$$

ii The sign of the constant term is positive, for example, $p^2 + 4 = 0$.
$p^2 + 4 = 0 \quad \Rightarrow \quad p^2 = -4$, so there is no real-valued solution.

Note

Depending on the calculator you are using, $\sqrt{(-4)}$ may be displayed as 'Math error' or '2i', where i is used to denote $\sqrt{(-1)}$. This is a **complex number** or **imaginary number** which you will meet if you study Further Mathematics at Advanced Level.

The vertical line of symmetry

Graphs of all quadratic functions have a vertical line of symmetry. You can use this to find the maximum or minimum value of the function. If the graph crosses the horizontal axis, then the line of symmetry is halfway between the two points of intersection. The maximum or minimum value lies on this line of symmetry.

Worked example

a Plot the graph of $y = x^2 - 4x - 5$ for values of x from -2 to $+6$.

b Identify the values of x where the curve intersects the horizontal axis.

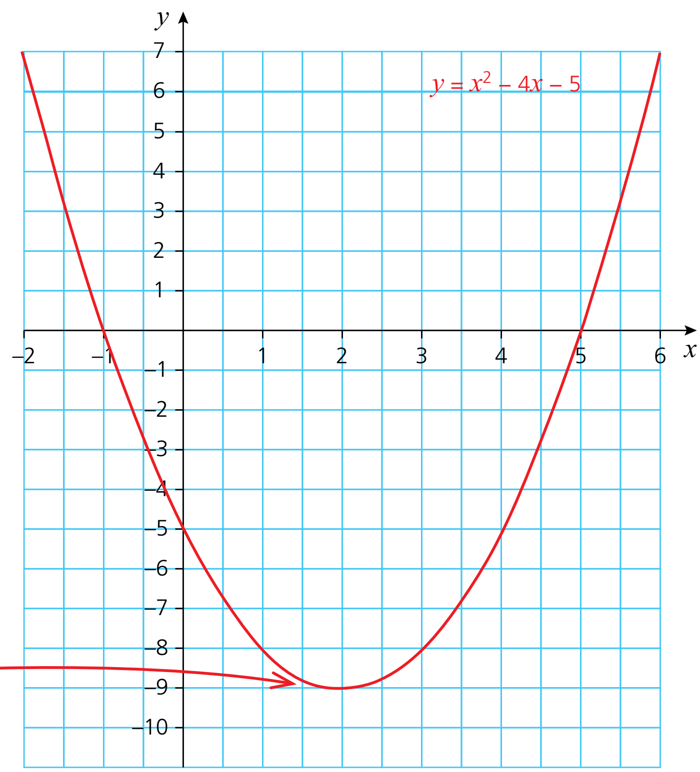
c Hence find the coordinates of the minimum or maximum point.

Solution

First create a table of values for $-2 \leqslant x \leqslant 6$.

a

x	–2	–1	0	1	2	3	4	5	6
y	7	0	–5	–8	–9	–8	–5	0	7



This point is often referred to as the **turning point** of the curve.

This is also shown in the table.

b The graph intersects the horizontal axis when $x = -1$ and when $x = 5$.

The line $x = 2$ passes through the turning point. It is a **vertical line of symmetry** for the curve.

c The graph shows that the curve has a minimum turning point halfway between $x = -1$ and $x = 5$. The table shows that the coordinates of this point are $(2, -9)$.

Factorising

Drawing graphs by hand to find maximum or minimum values can be time-consuming. The following example shows you how to use algebra to find these values.

Worked example

The first step is to factorise the expression. One method of factorising is shown, but if you are confident using a different method then continue to use it.

Find the coordinates of the turning point of the curve $y = x^2 + x - 6$. State whether the turning point is a maximum or a minimum value.

Solution

Find two integers (whole numbers) that multiply together to give the **constant** term, –6.

Possible pairs of numbers with a product of –6 are: 6 and –1, 1 and –6, 3 and –2, 2 and –3.

Identify any of the pairs of numbers that can be added together to give the coefficient of x (1). 3 and –2 are the only pair with a sum of 1, so use this pair to split up the x term.

Both expressions in the brackets must be the same. Notice the sign change due to the negative sign in front of the 2.

$$\begin{aligned} x^2 + x - 6 &= x^2 + 3x - 2x - 6 \\ &= x(x + 3) - 2(x + 3) \\ &= (x + 3)(x - 2) \end{aligned}$$

Note

You would get the same result if you used $3x$ and $-2x$ in the opposite order:

$$\begin{aligned} x^2 + x - 6 &= x^2 - 2x + 3x - 6 \\ &= x(x - 2) + 3(x - 2) \\ &= (x - 2)(x + 3) \end{aligned}$$

The graph of $y = x^2 + x - 6$ crosses the x-axis when $(x + 3)(x - 2) = 0$, i.e. when $x = -3$ and when $x = 2$.

The x-coordinate of the turning point is halfway between these two values, so:

$$\begin{aligned} x &= \frac{-3 + 2}{2} \\ &= -0.5 \end{aligned}$$

Substituting this value into the equation of the curve gives:

$$\begin{aligned} y &= (-0.5)^2 + (-0.5) - 6 \\ &= -6.25 \end{aligned}$$

The equation of the curve has a positive x^2 term so its graph is ∪-shaped. Therefore the minimum value is at (–0.5, –6.25).

The method shown above can be adapted for curves with an equation in which the coefficient of x^2 is not +1, for example, $y = 6x^2 - 13x + 6$ or $y = 6 - x - 2x^2$, as shown in the next example.

Worked example

For the curve with equation $y = 6 - x - 2x^2$:

a Will the turning point of the curve be a maximum or a minimum? Give a reason for your answer.

b Write down the coordinates of the turning point.

c State the equation of the line of symmetry.

Solution

a The coefficient of x^2 is negative so the curve will be ∩-shaped. This means that the turning point will be a maximum.

Continue to use any alternative methods of factorising that you are confident with.

b First multiply the constant term and the coefficient of x^2, i.e. $6 \times -2 = -12$. Then find two whole numbers that multiply together to give this product.

Possible pairs are: 6 and −2, −6 and 2, −3 and 4, −3 and 4, 1 and −12, −1 and 12.

Identify any of the pairs of numbers that can be added together to give the coefficient of x (−1).

−3 and 4 are the only pair with a sum of −1, so use this pair to split up the x term.

$$\begin{aligned} 6 - x - 2x^2 &= 6 + 3x - 4x - 2x^2 \\ &= 3(2 + x) - 2x(2 + x) \\ &= (2 + x)(3 - 2x) \end{aligned}$$

Both expressions in the brackets must be the same. Notice the sign change is due to the sign in front of the 2.

The graph of $y = 6 - x - 2x^2$ crosses the x-axis when $(2 + x)(3 - 2x) = 0$, i.e. when $x = -2$ and when $x = 1.5$.

The x-coordinate of the turning point is halfway between these two values.

$$\begin{aligned} x &= \frac{-2 + 1.5}{2} \\ &= -0.25 \end{aligned}$$

Substituting this value into the equation of the curve gives:

$$\begin{aligned} y &= 6 - (-0.25) - 2(-0.25)^2 \\ &= 6.125 \end{aligned}$$

So the turning point is (−0.25, 6.125)

c The equation of the line of symmetry is $x = -0.25$.

Completing the square

The methods shown in the previous examples will always work for curves that cross the x-axis. For quadratic curves that do not cross the x-axis, you will need to use the method of **completing the square**, shown in the next example.

Another way of writing the quadratic expression $x^2 + 6x + 11$ is $(x + 3)^2 + 2$ and this is called **completed square form**. Written like this the expression consists of a squared term, $(x + 3)^2$, that includes the variable, x, and a constant term +2.

In the next example you see how to convert an ordinary quadratic expression into completed square form.

Worked example

a Write $x^2 - 8x + 18$ in completed square form.

b State whether it is a maximum or minimum.

c Sketch the curve $y = f(x)$.

Solution

a Start by halving the coefficient of x and squaring the result.

$-8 \div 2 = -4$
$(-4)^2 = 16$

Now use this result to break up the constant term, +18, into two parts:

$18 = 16 + 2$

and use this to rewrite the original expression as:

You will always have a perfect square in this expression.

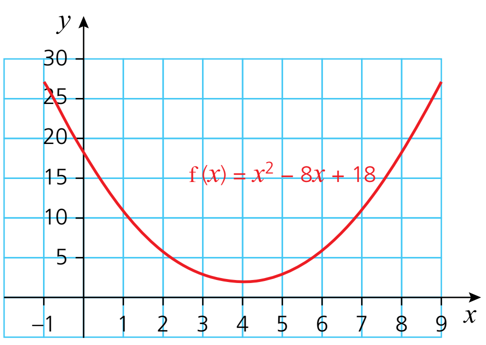
$f(x) = x^2 - 8x + 16 + 2$
$= (x - 4)^2 + 2$
$(x - 4)^2 \geqslant 0$ (always)
$\Rightarrow f(x) \geqslant 2$ for all values of x

In completed square form, $x^2 - 8x + 18 = (x - 4)^2 + 2$

b $f(x) \geqslant 2$ for all values of x so the turning point is a minimum.

c The function is a ∪-shaped curve because the coefficient of x^2 is positive. From the above, the minimum turning point is at (4, 2) so the curve does not cross the x-axis. To sketch the graph, you will also need to know where it crosses the y-axis.

$f(x) = x^2 - 8x + 18$ crosses the y-axis when $x = 0$, i.e. at (0, 18).



Worked example

Use the method of completing the square to work out the coordinates of the turning point of the quadratic function $f(x) = 2x^2 - 8x + 9$.

Solution

$f(x) = 2x^2 - 8x + 9$
$= 2(x^2 - 4x) + 9$
$= 2((x - 2)^2 - 4) + 9$
$= 2(x - 2)^2 + 1$

$(x-2)^2 \geqslant 0$ (always), so the minimum value of $f(x)$ is 1.

When $f(x) = 1$, $x = 2$.

Therefore the coordinates of the turning point (minimum value) of the function $f(x) = 2x^2 - 8x + 9$ are (2, 1).

Sometimes you will be asked to sketch the graph of a function $f(x)$ for certain values of x. This set of values of x is called the **domain** of the function. The corresponding set of y-values is called the **range**.

Worked example

The domain of the function $y = 6x^2 + x - 2$ is $-3 \leqslant x \leqslant 3$.

Sketch the graph and find the range of the function.

Solution

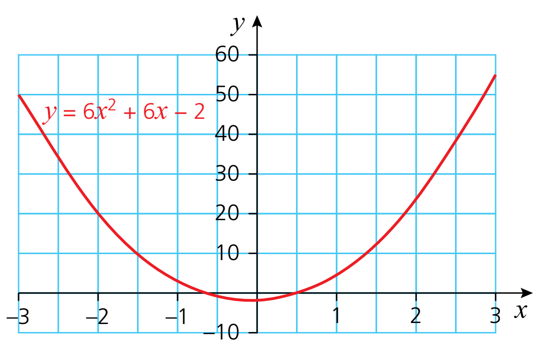
The coefficient of x^2 is positive, so the curve is ∪-shaped and the turning point is a minimum.

The curve crosses the x-axis when $6x^2 + x - 2 = 0$.

$$6x^2 + x - 2 = (3x+2)(2x-1)$$
$$\Rightarrow (3x+2)(2x-1) = 0$$
$$\Rightarrow \quad (3x+2) = 0 \text{ or } (2x-1) = 0$$

So the graph crosses the x-axis at $(-\frac{2}{3}, 0)$ and $(\frac{1}{2}, 0)$.

The curve crosses the y-axis when $x = 0$, i.e. at $(0, -2)$.



The curve has a vertical line of symmetry passing halfway between the two points where the curve intersects the x-axis. Therefore the equation of this line of symmetry is $x = \dfrac{-\frac{2}{3} + \frac{1}{2}}{2}$ or $x = -\frac{1}{12}$.

When $x = -\frac{1}{12}$, $y = 6\left(-\frac{1}{12}\right)^2 + \left(-\frac{1}{12}\right) - 2 = -2\frac{1}{24}$, the minimum value of the function.

To find the range, work out the values of y for $x = -3$ and $x = +3$.

The larger of these gives the maximum value.

When $x = -3$, $y = 6(-3)^2 + (-3) - 2 = 49$.
When $x = 3$, $y = 6(3)^2 + 3 - 2 = 55$.

The range of the function corresponding to the domain $-3 \leqslant x \leqslant 3$ is therefore $-2\frac{1}{24} \leqslant y \leqslant 55$.

Exercise 2.1

1 Solve each equation by factorising:

a $x^2 + x - 20 = 0$ **b** $x^2 - 5x + 6 = 0$

c $x^2 - 3x - 28 = 0$ **d** $x^2 + 13x + 42 = 0$

2 Solve each equation by factorising:

a $2x^2 - 3x + 1 = 0$ **b** $9x^2 + 3x - 2 = 0$

c $2x^2 - 5x - 7 = 0$ **d** $3x^2 + 17x + 10 = 0$

3 Solve each equation by factorising:

a $x^2 - 169 = 0$ **b** $4x^2 - 121 = 0$

c $100 - 64x^2 = 0$ **d** $12x^2 - 27 = 0$

4 For each of the following curves:

i Factorise the function.

ii Work out the coordinates of the turning point.

iii State whether the turning point is a maximum or minimum.

iv Sketch the graph, labelling the coordinates of the turning point and any points of intersection with the axes.

a $y = x^2 + 7x + 10$ **b** $f(x) = 16 - 6x - x^2$

c $y = 5 - 9x - 2x^2$ **d** $f(x) = 2x^2 + 11x + 12$

5 Write each quadratic expressions in the form $(x + a)^2 + b$:

a $x^2 + 4x + 9$ **b** $x^2 - 10x - 4$

c $x^2 + 5x - 7$ **d** $x^2 - 9x - 2$

6 Write each quadratic expression in the form $c(x + a)^2 + b$.

a $2x^2 - 12x + 5$ **b** $3x^2 + 12x + 20$

c $4x^2 - 8x + 5$ **d** $2x^2 + 9x + 6$

7 Solve the following quadratic equations. Leave your answers in the form $x = p \pm \sqrt{q}$.

a $x^2 + 4x - 9 = 0$ **b** $x^2 - 7x - 2 = 0$

c $2x^2 + 6x - 9 = 0$ **d** $3x^2 + 9x - 15 = 0$

8 For each of the following functions:

i Use the method of completing the square to find the coordinates of the turning point of the graph.

ii State whether the turning point is a maximum or a minimum.

iii Sketch the graph.

a $f(x) = x^2 + 6x + 15$ **b** $y = 8 + 2x - x^2$

c $y = 2x^2 + 2x - 9$ **d** $f : x \rightarrow x^2 - 8x + 20$

9 Sketch the graph and find the corresponding range for each function and domain.

a $y = x^2 - 7x + 10$ for the domain $1 \leqslant x \leqslant 6$

b $f(x) = 2x^2 - x - 6$ for the domain $-2 \leqslant x \leqslant 2$

Real-world activity

1 Draw a sketch of a bridge modelled on the equation $25y = 100 - x^2$ for $-10 \leqslant x \leqslant 10$. Label the origin O, point A(−10, 0), point B(10, 0) and point C(0, 4).

2 1 unit on your graph represents 1 metre. State the maximum height of the bridge, OC, and the span, AB.

3 Work out the equation of a similar bridge with a maximum height of 5 m and a span of 40 m.

The quadratic formula

The **roots** of a quadratic equation f(x) are those values of x for which $y = 0$ for the curve $y = $ f(x). In other words, they are the x-coordinates of the points where the curve either crosses or touches the x-axis. There are three possible outcomes.

1 The curve crosses the x-axis at two distinct points. In this case, the corresponding equation is said to have *two real distinct roots.*
2 The curve touches the x-axis, in which case the equation has *two equal (repeating) roots.*
3 The curve lies completely above or completely below the x-axis so it neither crosses nor touches the axis. In this case, the equation has *no real roots.*

The method of completing the square can be generalised to give a formula for solving quadratic equations. The next example uses this method in a particular case on the left-hand side and shows the same steps for the general case on the right-hand side, using algebra to derive the formula for solving quadratic equations.

Worked example

Solve $2x^2 + x - 4 = 0$.

Solution

$$2x^2 + x - 4 = 0$$
$$\Rightarrow \quad x^2 + \frac{1}{2}x - 2 = 0$$
$$\Rightarrow \quad x^2 + \frac{1}{2}x = 2$$
$$\Rightarrow \quad x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)x^2 = 2 + \left(\frac{1}{4}\right)^2$$
$$\Rightarrow \quad \left(x + \frac{1}{4}\right)^2 = \frac{33}{16}$$
$$\Rightarrow \quad \left(x + \frac{1}{4}\right) = \pm\frac{\sqrt{33}}{4}$$
$$\Rightarrow \quad x = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}$$
$$= \frac{-1 \pm \sqrt{33}}{4}$$

Generalisation

$$ax^2 + bx + c = 0$$
$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$
$$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$\Rightarrow \quad x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$= \frac{b^2 - 4ac}{4a^2}$$
$$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$
$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The result $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as the **quadratic formula**. You can use it to solve any quadratic equation. One root is found by taking the + sign, and the other by taking the – sign. When the value of $b^2 - 4ac$ is negative, the square root cannot be found and so there is no real solution to that quadratic equation. This occurs when the curve does not cross the x-axis.

In an equation of the form $(px + q)^2 = 0$, where p and q can represent either positive or negative numbers, $px + q = 0$ gives the only solution.

Note

The part $b^2 - 4ac$ is called the **discriminant** because it discriminates between quadratic equations with no roots, quadratic equations with one repeated root and quadratic equations with two real roots.

- If $b^2 - 4ac > 0$ there are 2 real roots.
- If $b^2 - 4ac = 0$ there is 1 repeated root.
- If $b^2 - 4ac < 0$ there are no real roots.

Worked example

a Show that the equation $4x^2 - 12x + 9 = 0$ has a repeated root by:

  i factorising **ii** using the discriminant.

b State with reasons how many real roots the following equations have:

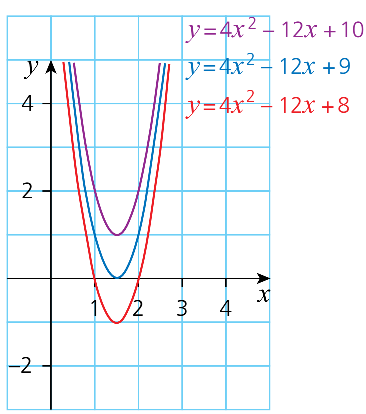
  i $4x^2 - 12x + 8 = 0$ **ii** $4x^2 - 12x + 10 = 0$

Solution

a i

$$\begin{aligned} 4x^2 - 12x + 9 &= 0 \\ \Rightarrow (2x-3)(2x-3) &= 0 \\ 2x - 3 &= 0 \\ \Rightarrow x &= 1.5 \end{aligned}$$

ii The equation has a repeated root because the discriminant $b^2 - 4ac = (-12)^2 - 4(4)(9) = 0$.



b i The curve $y = 4x^2 - 12x + 8$ is 1 unit below $y = 4x^2 - 12x + 9$ and crosses the x-axis in 2 points. So the equation has two real roots.

ii The curve $y = 4x^2 - 12x + 10$ is 1 unit above $y = 4x^2 - 12x + 9$ and does not cross the x-axis. So the equation $4x^2 - 12x + 10 = 0$ has no real roots.

In some cases, such as in the previous example, the factorisation is not straightforward. In such cases, evaluating the discriminant is a reliable method to obtain an accurate result.

Worked example

Show that the equation $3x^2 - 2x + 4 = 0$ has no real solution.

Solution

The most straightforward method is to look at the discriminant. If the discriminant is negative, there is no real solution.

For $3x^2 - 2x + 4 = 0$, $a = 3$, $b = -2$ and $c = 4$.

Substituting these values into the discriminant

$$b^2 - 4ac = (-2)^2 - 4(3)(4)$$
$$= -44$$

Since the discriminant is negative, there is no real solution.

The general equation of a straight line is $y = mx + c$. This has alternate forms, e.g. $ax + by + c = 0$.

The intersection of a line and a curve

The examples so far have considered whether or not a curve intersects, touches, or lies completely above or below the x-axis ($y = 0$). The next example considers the more general case of whether or not a curve intersects, touches or lies completely above or below a particular straight line.

Worked example

a Find the coordinates of the points of intersection of the line $y = 4 - 2x$ and the curve $y = x^2 + x$.

b Sketch the line and the curve on the same axes.

Solution

The y-values of both equations are the same at the point(s) of intersection.

a To find where the curve and the lines intersect, solve $y = x^2 + x$ simultaneously with $y = 4 - 2x$.

$$x^2 + x = 4 - 2x$$
$$\Rightarrow \quad x^2 + 3x - 4 = 0$$
$$\Rightarrow \quad (x + 4)(x - 1) = 0$$
$$\Rightarrow \quad x = -4 \text{ or } x = 1$$

It is more straightforward to substitute into the linear equation.

To find the y-coordinate, substitute into one of the equations.

When $x = -4$, $y = 4 - 2(-4) = 12$.
When $x = 1$, $y = 4 - 2(1) = 2$.
The line $y = 4 - 2x$ intersects the curve $y = x^2 + x$ at $(-4, 12)$ and $(1, 2)$.

b The curve has a positive coefficient of x^2 so is ∪-shaped.

It crosses the x-axis when $x^2 + x = 0$.

$\Rightarrow \quad x(x + 1) = 0$

$\Rightarrow \quad x = 0$ or $x = -1$

So the curve crosses the x-axis at $x = 0$ and $x = -1$.

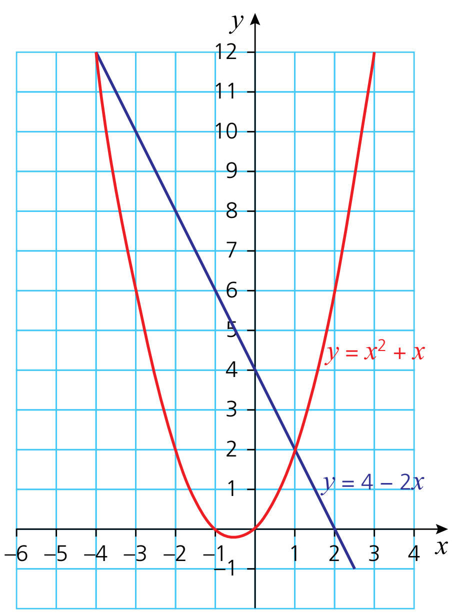
It crosses the y-axis when $x = 0$.

Substituting $x = 0$ into $y = x^2 + x$ gives $y = 0$.

So the curve passes through the origin.

The line $2x + y = 4$ crosses the x-axis when $y = 0$. When $y = 0$, $x = 2$.

The line $2x + y = 4$ crosses the y-axis when $x = 0$. When $x = 0$, $y = 4$.



It is possible for a quadratic curve to touch a general line, either sloping or parallel to the x-axis. You can see this when you solve the equations of the line and the curve simultaneously. If you get a repeated root, it means that they touch at only one point. The line is a tangent to the curve. This is shown in the next example.

Worked example

a Use algebra to show that the line $y = 6x - 19$ touches the curve $y = x^2 - 2x - 3$ and find the coordinates of the point of contact.

b Sketch the line and curve on the same axes.

Solution

a Solving the equations simultaneously

$$x^2 - 2x - 3 = 6x - 19$$
$$\Rightarrow \quad x^2 - 8x + 16 = 0$$
$$\Rightarrow \quad (x-4)^2 = 0$$
$$\Rightarrow \quad x = 4$$

The repeated root $x = 4$ shows that the line and the curve touch.

Substitute $x = 4$ into either equation to find the value of the y-coordinate.

It is more straightforward to substitute into the line equation.

$$y = 6(4) - 19$$
$$= 5$$

Therefore the point of contact is (4, 5).

b The coefficient of x^2 is positive so the curve is ∪-shaped.

Substituting $x = 0$ into $y = x^2 - 2x - 3$ shows that the curve intersects the y-axis at (0, –3).

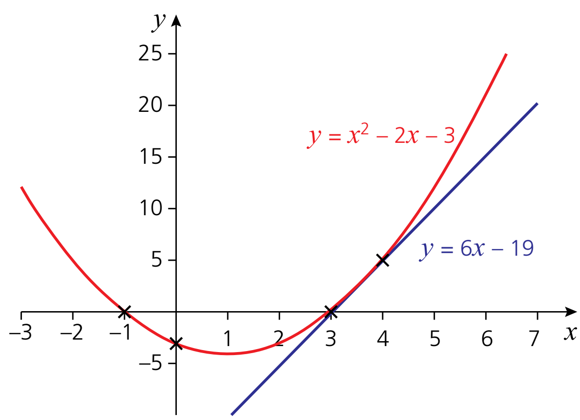
Substituting $y = 0$ into $y = x^2 - 2x - 3$ gives $x^2 - 2x - 3 = 0$.

$\Rightarrow \quad (x-3)(x+1) = 0$

$\Rightarrow \quad x = -1$ or $x = 3$

So the curve intersects the x-axis at (–1, 0) and (3, 0).

You need two points to draw a line. It is best to choose points with whole numbers that are not too large, such as (3, –1) and (5, 11).



Discussion point

Why is it not possible for a quadratic curve to touch a line parallel to the y-axis?

There are many situations when a line and a curve do not intersect or touch each other. A straightforward example of this occurs when the graph of a quadratic function is a ∪-shaped curve completely above the x-axis, e.g. $y = x^2 + 3$, and the line is the x-axis.

You have seen how solving the equations of a curve and a line simultaneously gives a quadratic equation with *two* roots when the line crosses the curve, and a quadratic equation with a *repeated* root when it touches the curve. If solving the two equations simultaneously results in *no* real roots, i.e. the discriminant is negative, then they do not cross or touch.

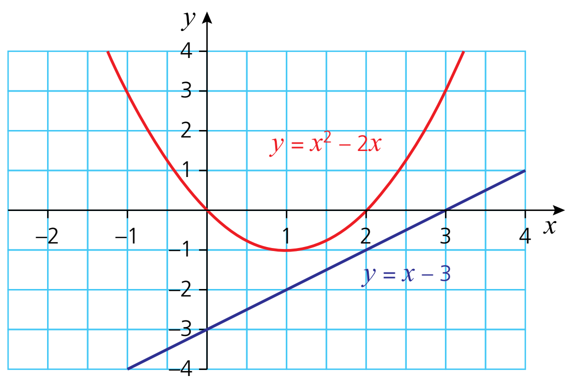
Worked example

a Sketch the graphs of the line $y = x - 3$ and the curve $y = x^2 - 2x$ on the same axes.

b Use algebra to prove that the line and the curve don't meet.

Solution

a



b $x^2 - 2x = x - 3$ ← Solving the two equations simultaneously

$\Rightarrow x^2 - 3x + 3 = 0$

This does not factorise, so solve using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1$, $b = -3$ and $c = 3$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{-3}}{2}$$

You do not actually need to write down the solution. Once you see that the value of the discriminant is negative, as in this case where it is –3, you know that the equation has no real roots, so the line and the curve don't meet.

Since there is a negative value under the square root, there is no *real* solution. This implies that the line and the curve do not meet.

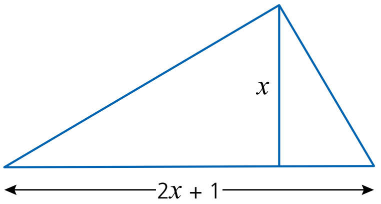
Note

It would have been sufficient to consider only the discriminant $b^2 - 4ac$. Solving a quadratic equation is equivalent to finding the point(s) where the curve crosses the horizontal axis (the roots).

Using quadratic equations to solve problems

Worked example

A triangle has a base of $(2x + 1)$ cm, a height of x cm and an area of 68 cm^2.



a Show that x satisfies the equation $2x^2 + x - 136 = 0$.

b Solve the equation and work out the base length of the triangle.

Solution

a Using the formula for the area of a triangle, area $= \frac{1}{2}$base × height:

$$\text{Area} = \frac{1}{2} \times (2x + 1) \times x$$

$$= \frac{1}{2}(2x^2 + x)$$

The area is 68 cm^2, so:

$$\frac{1}{2}(2x^2 + x) = 68$$

$$\Rightarrow \quad 2x^2 + x = 136$$

$$\Rightarrow \quad 2x^2 + x - 136 = 0$$

Alternatively, you can use a calculator to solve the equation.

b It is not easy to factorise this equation – it is not even obvious that there are factors – so use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 2$, $b = 1$ and $c = -136$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-136)}}{2(2)}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1089}}{4}$$

$$\Rightarrow x = \frac{-1 \pm 33}{4}$$

$$\Rightarrow x = 8 \text{ or } x = -8.5$$

Since x is a length, reject the negative solution.

Substitute $x = 8$ into the expression for the base of the triangle, $2x + 1$, and work out the length of the base of the triangle, 17 cm.

Check that this works with the information given in the original question.

$$\frac{1}{2} \times 17\text{ cm} \times 8\text{ cm} = 68\text{ cm}^2$$

Solving quadratic inequalities

The quadratic inequalities in this section all involve quadratic expressions that factorise. This means that you can find a solution either by sketching the appropriate graph or by using line segments to reduce the quadratic inequality to two simultaneous linear inequalities.

The example below shows two valid methods for solving quadratic inequalities. You should use whichever method you prefer. Your choice may depend on how easily you sketch graphs or if you have a graphic calculator that you can use to plot these graphs.

Worked example

Solve these quadratic inequalities.

a $x^2 - 2x - 3 < 0$

b $x^2 - 2x - 3 \geqslant 0$

Solution

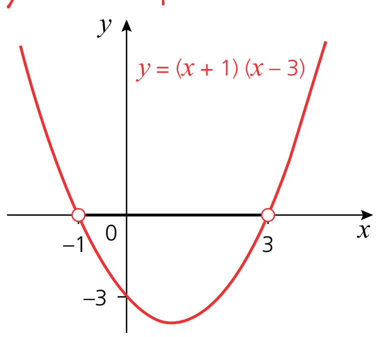
Method 1

$x^2 - 2x - 3 = (x + 1)(x - 3)$

So the graph of $y = x^2 - 2x - 3$ crosses the x-axis when $x = -1$ and $x = 3$.

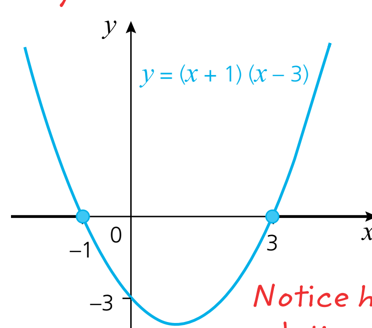
Look at the two graphs below.

Here the end points are not included in the solution, so you draw open circles: ○



The solution is $-1 < x < 3$.

Here the end points are included in the solutions, so you draw solid circles: ●



The solution is $x \leqslant -1$ or $x \geqslant 3$.

Notice how the solution is in two parts when there are two line segments.

a The answer is the values of x for which $y < 0$, i.e. where the curve is below the x-axis.

b The answer is the values of x for which $y \geqslant 0$, i.e. where the curve crosses or is above the x-axis.

Method 2

This method identifies the values of x for which each of the factors is 0 and considers the sign of each factor in the intervals between these critical values.

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
Sign of $(x + 1)$	–	0	+	+	+
Sign of $(x - 3)$	–	–	–	0	+
Sign of $(x + 1)(x - 3)$	$(-) \times (-) = +$	$(0) \times (-) = 0$	$(+) \times (-) = -$	$(+) \times (0) = 0$	$(+) \times (+) = +$

From the table, the solution to:

a $(x + 1)(x - 3) < 0$ is $-1 < x < 3$

b $(x + 1)(x - 3) \geqslant 0$ is $x \leqslant -1$ or $x \geqslant 3$

If the inequality to be solved contains $>$ or $<$, then the solution is described using $>$ and $<$. If the original inequality contains $\geqslant$ or $\leqslant$, then the solution is described using $\geqslant$ and $\leqslant$.

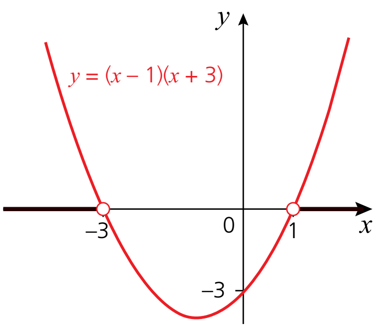
If the quadratic inequality has the variable on both sides, collect the terms involving the variable on one side first in the same way as you would before solving a quadratic equation.

Worked example

Solve $2x + x^2 > 3$.

Solution

$$2x + x^2 > 3 \Rightarrow x^2 + 2x - 3 > 0$$
$$\Rightarrow (x - 1)(x + 3) > 0$$



From the graph, the solution is $x < -3$ or $x > 1$.

Exercise 2.2

1 For each of the following equations, decide if there are two real and different roots, two equal roots or no real roots. Solve the equations with real roots.

a $x^2 + 3x + 2 = 0$
b $t^2 - 9 = 0$
c $x^2 + 16 = 0$
d $2x^2 - 5x = 0$
e $p^2 + 3p - 18 = 0$
f $x^2 + 10x + 25 = 0$
g $15a^2 + 2a - 1 = 0$
h $3r^2 + 8r = 3$

2 Solve the following equations by:

i completing the square ii using the quadratic formula.

Give your answers correct to two decimal places.

a $x^2 - 2x - 10 = 0$ b $x^2 + x = 0$
c $2x^2 + 2x - 9 = 0$ d $2x^2 + x - 8 = 0$

3 Try to solve each of the following equations. Where there is a solution, give your answers correct to two decimal places.

a $4x^2 + 6x - 9 = 0$ b $9x^2 + 6x + 4 = 0$
c $(2x + 3)^2 = 7$ d $x(2x - 1) = 9$

4 Use the discriminant to decide whether each of the following equations has two equal roots, two distinct roots or no real roots:

a $9x^2 - 12x + 4 = 0$ b $6x^2 - 13x + 6 = 0$ c $2x^2 + 7x + 9 = 0$
d $2x^2 + 9x + 10 = 0$ e $3x^2 - 4x + 5 = 0$ f $4x^2 + 28x + 49 = 0$

5 For each pair of equations determine if the line intersects the curve, is a tangent to the curve or does not meet the curve. Give the coordinates of any points where the line and curve touch or intersect.

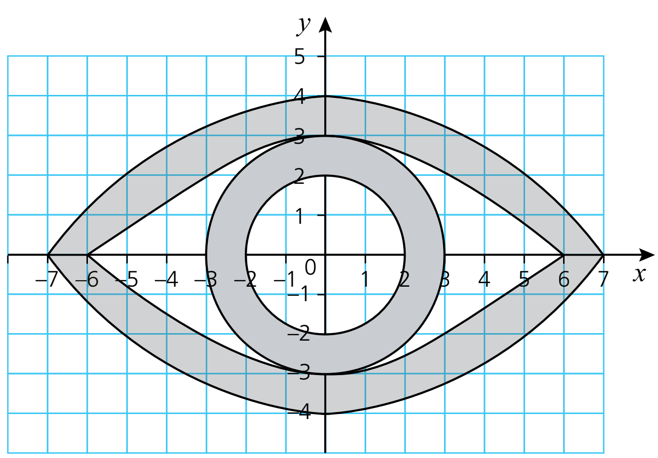
a $y = x^2 + 12x; \quad y = 9 + 6x$ b $y = 2x^2 + 3x - 4; \quad y = 2x - 6$
c $y = 6x^2 - 12x + 6; \quad y = x$ d $y = x^2 - 8x + 18; \quad y = 2x + 3$
e $y = x^2 + x; \quad 2x + y = 4$ f $y = 4x^2 + 9; \quad y = 12x$
g $y = 3 - 2x - x^2; \quad y = 9 + 2x$ h $y = (3 - 2x)^2; \quad y = 2 - 3x$

6 Solve the following inequalities and illustrate each solution on a number line:

a $x^2 - 6x + 5 > 0$ b $a^2 + 3a - 4 \leqslant 0$ c $4 - y^2 > 0$
d $x^2 - 4x + 4 > 0$ e $8 - 2a > a^2$ f $3y^2 + 2y - 1 > 0$

Real-world activity

Anna would like to design a pendant for her mother and decides that it should resemble an eye. She starts by making the scale drawing, shown below.



The pendant is made up of the shaded area.

The equations of the two circles are $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

The rest of the pendant is formed by quadratic curves.

The scale is 2 units represents 1 cm.

1 Find the equations of the four quadratic curves.
2 Anna decides to make some earrings using a smaller version of the pendant design. She reduces the size by a factor of 2. Find the equations of the four quadratic curves for the earrings.

Past-paper questions

1 (i) Express $2x^2 - x + 6$ in the form $p(x - q)^2 + r$, where p, q and r are constants to be found. [3]

(ii) Hence state the least value of $2x^2 - x + 6$ and the value of x at which this occurs. [2]

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2 Find the set of values of k for which the curve $y = 2x^2 + kx + 2k - 6$ lies above the x-axis for all values of x. [4]

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3 The line $y = mx + 2$ is a tangent to the curve $y = x^2 + 12x + 18$. Find the possible values of m. [4]

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Learning outcomes

Now you should be able to:

- ★ find the maximum or minimum value of the quadratic function f: $x \mapsto ax^2 + bx + c$ by any method
- ★ use the maximum or minimum values of f(x) to sketch the graph or determine the range for a given domain
- ★ know the conditions for f(x) = 0 to have two real roots, two equal roots or no real roots and know the related conditions for a given line to intersect a given curve, be a tangent to a given curve or not intersect a given curve
- ★ solve quadratic equations for real roots and find the solution set for quadratic inequalities.

Key points

- ✔ A **quadratic function** has the form $f(x) = ax^2 + bx + c$, where a, b and c can be any number (positive, negative or zero) provided that $a \neq 0$. The set of possible values of x is called the **domain** of the function and the set of y values is called the **range**.
- ✔ To plot the graph of a quadratic function, first calculate the value of y for each value of x in the given range.
- ✔ The graph of a quadratic function is symmetrical about a vertical line. It is ∪-shaped if the coefficient of x^2 is positive and ∩-shaped if the coefficient of x^2 is negative.
- ✔ To sketch the graph of a quadratic function:
  - look at the coefficient of x^2 to determine the shape
  - substitute $x = 0$ to determine where the curve crosses the vertical axis
  - solve $f(x) = 0$ to determine any values of x where the curve touches or crosses the horizontal axis.
- ✔ If there are no real values for x for which $f(x) = 0$, then the curve will be either completely above or completely below the x-axis.
- ✔ A **quadratic equation** is of the form $ax^2 + bx + c$ with $a \neq 0$.
- ✔ To **factorise** a quadratic equation of the form $x^2 + bx + c = 0$, look for two numbers, p and q, with the sum b and the product c. The factorised form is then $(x - p)(x - q) = 0$. To factorise an equation of the form $ax^2 + bx + c = 0$, look for two numbers with the sum b and the product ac.

- ✔ The **discriminant** of a quadratic equation ($ax^2 + bx + c = 0$) is $b^2 - 4ac$. If $b^2 - 4ac > 0$, a quadratic equation will have two distinct solutions (or roots). If $b^2 - 4ac = 0$, the two roots are equal so there is one repeating root. If $b^2 - 4ac < 0$, the roots have no real values.
- ✔ An expression of the form $(px + q)^2$ is called a **perfect square**.
- ✔ $x^2 + bx + c$ can be written as $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$ using the method of **completing the square**. For expressions of the form $ax^2 + bx + c$, first take a out as a factor.
- ✔ The **quadratic formula** for solving an equation of the form $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- ✔ To find the point(s) where a line and a curve touch or intersect, substitute the expression for y from one equation into the other to give a quadratic equation in x.
- ✔ When solving a **quadratic inequality**, it is advisable to start by sketching the associated quadratic graph.