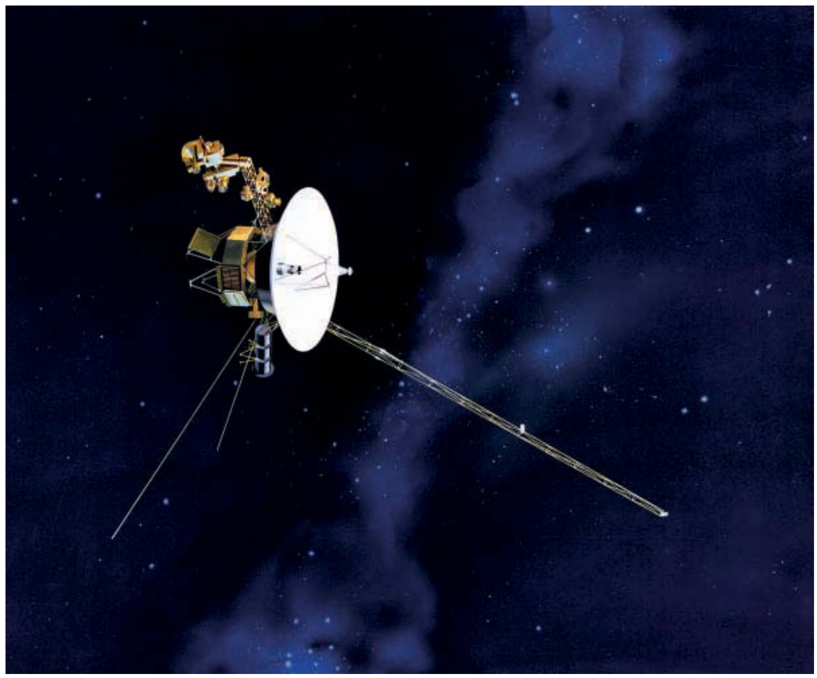
16 Kinematics

In future, children won't perceive the stars as mere twinkling points of light: they'll learn that each is a Sun, orbited by planets fully as interesting as those in our Solar system.

Martin Rees (1942 –)



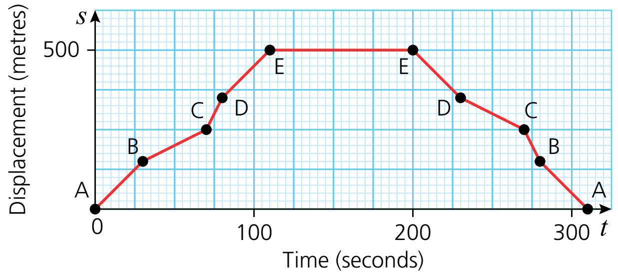
Discussion point

A spacecraft leaves the Earth on a journey to Jupiter. Its initial direction is directly towards Jupiter. Will it travel in a straight line?

Displacement and velocity

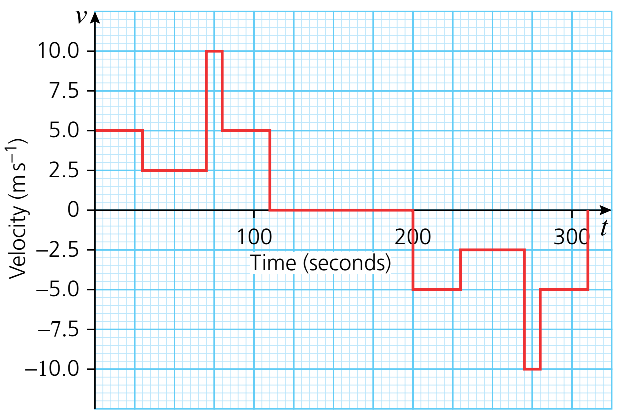
You know that $\frac{\mathrm{d}y}{\mathrm{d}x}$ represents the rate of change of y with respect to x. It gives the gradient of the x–y graph, where x is plotted on the horizontal axis and y on the vertical axis.

The following graph represents the distance, s metres, travelled by a cyclist along a country road in time, t seconds. Time is measured along the horizontal axis and distance from the starting point is measured on the vertical axis. When he reaches E the cyclist takes a short break and then returns home along the same road.



Speed is given by the gradient of the distance–time graph. In this graph the axes are labelled s and t, rather than y and x, so the gradient (representing the speed) is given by $\frac{ds}{dt}$.

A graph showing **displacement** (the distance from the starting point) looks quite different from one showing the total distance travelled.



Velocity is given by the gradient of a displacement–time graph.

Acceleration is $\left(\frac{\text{change in velocity}}{\text{time taken}}\right)$ and this is given by the gradient of a velocity–time graph.

Motion in a straight line

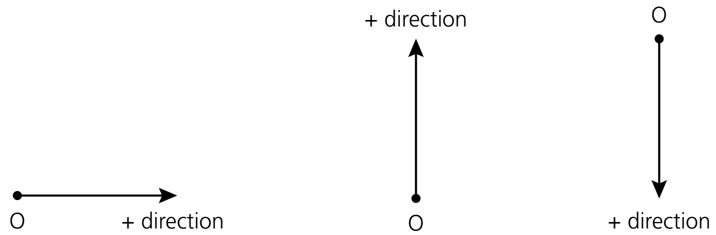
You will be treating each object as a **particle**, i.e. something with a mass but no dimension.

In the work that follows you will use displacement, which measures position, rather than distance travelled.

Before doing anything else you need to make two important decisions:

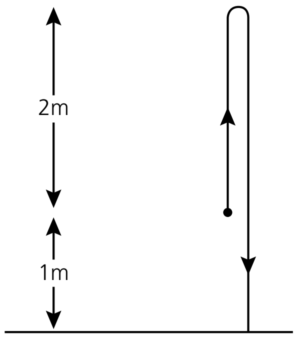
1 where you will take your origin
2 which direction you will take as positive.

Some options are:

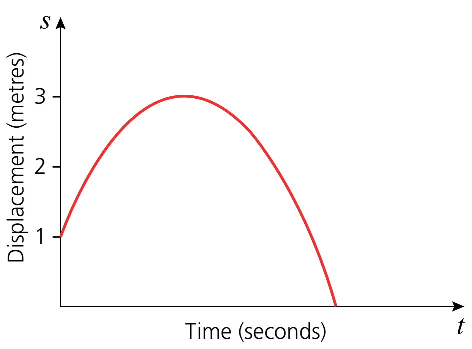


Think about the motion of a tennis ball that is thrown up vertically and allowed to fall to the ground, as in the diagram below. Assume that the ball leaves your hand at a height of 1 m above the ground and rises a further 2 m to the highest point. At this point the ball is *instantaneously at rest*.

This means it is about to change direction through 180°.



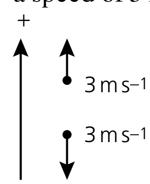
The displacement–time graph of the ball's flight is shown below. For this graph, displacement is measured from ground level with upwards as the positive direction.



Note

Be careful not to confuse the terms velocity and speed. Speed has magnitude (size) but no direction. Velocity has direction and magnitude. For example, taking upwards as the positive direction,

- a speed of 3 m s^{-1} upwards is a velocity +3 m s^{-1}
- a speed of 3 m s^{-1} downwards is a velocity of –3 m s^{-1}.



The table gives the terms that you will be using, together with their definitions, units and the letters that are commonly used to represent those quantities.

Quantity	Definition	S.I. unit	Unit symbol	Notation
Time	Measured from a fixed origin	second	s	t
Distance	Distance travelled in a given time	metre	m	x, y, s
Speed	Rate of change of distance	metre per second	m s^{-1}	$v = \frac{dx}{dt}$ etc.
Displacement	Distance from a fixed origin	metre	m	x, y, s, h
Velocity	Rate of change of displacement	metre per second	m s^{-1}	$v = \frac{ds}{dt}$
Acceleration	Rate of change of velocity	metre per second per second	m s^{-2}	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ etc.

Like velocity, acceleration can also be either positive or negative. A negative acceleration is an object either moving in the positive direction and slowing down, or moving in the negative direction and speeding up.

Motion with variable acceleration: the general case

If the motion involves variable acceleration, you must use calculus. You should know and be able to use these relationships.

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

These relationships are used in the next two examples.

Worked example

a The displacement in metres, s, of a sports car from its initial position during the first 4 seconds is given by

$$s = 12t^2 - t^3.$$

Find:

i an expression for the velocity in terms of t
ii the initial velocity
iii the velocity after 4 seconds
iv an expression for the acceleration in terms of t
v the accelerations after 4 seconds.

b The national speed limit in Great Britain is 70 mph.
At the end of 4 seconds, would the driver of this sports car be breaking the British national speed limit?

Solution

a i $v = \frac{ds}{dt}$

$= 24t - 3t^2$

ii When $t = 0$, $v = 0$

The initial velocity is $0\,\text{m s}^{-1}$.

iii When $t = 4$, $v = 24 \times 4 - 3 \times 4^2$

$= 48$

The velocity, after 4 seconds is $48\,\text{m s}^{-1}$.

iv $a = \frac{dv}{dt}$

$= 24 - 6t$

v When $t = 4$, $a = 24 - 6 \times 4$

$= 0$

The acceleration after 4 seconds is $0\,\text{m s}^{-2}$.

b $48\,\text{m s}^{-1} = \frac{48 \times 60 \times 60}{1000}$

$= 172.8\,\text{km h}^{-1}$

$172.8\,\text{km h}^{-1} \approx \frac{5}{8} \times 172.8$

$= 108\,\text{mph}$

The driver would be breaking the British speed limit.

Worked example

A particle travels in a straight line such that t seconds after passing through a fixed point O, its displacement s metres is given by $s = 5 + 2t^3 - 3t^2$.

a Find:

i expressions for the velocity and acceleration in terms of t
ii the times when it is at rest.

b Sketch the velocity–time graph.

c Find:
i how far it is from O when it is at rest
ii the initial acceleration of the particle.

Solution

a i $v = \frac{ds}{dt} = 6t^2 - 6t$

$a = \frac{dv}{dt} = 12t - 6$

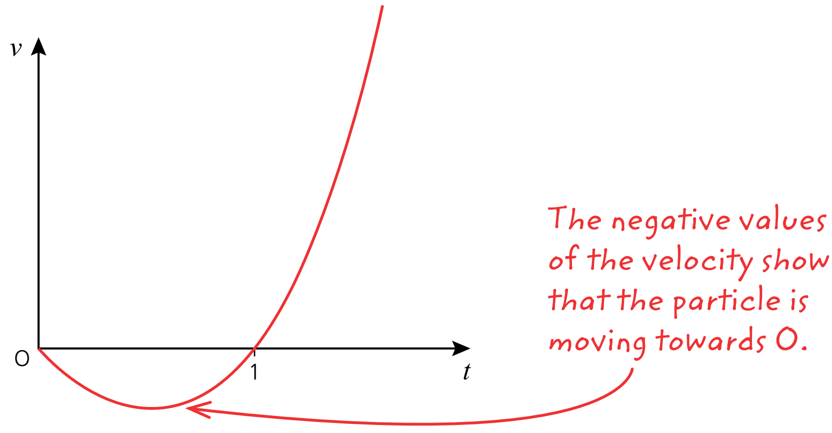
Notice that the acceleration varies with time.

ii The particle is at rest when $v = 0$.

$\Rightarrow \quad 6t^2 - 6t = 0$
$\Rightarrow \quad 6t(t - 1) = 0$
$\Rightarrow \quad t = 0$ or $t = 1$

So the particle is at rest initially and after 1 second.

b The graph of v against t is a ∪-shaped curve that crosses the t axis at $t = 0$ and $t = 1$.



c i When $t = 0$, $s = 5$.

When $t = 1$, $s = 5 + 2 - 3$
$= 4$.

The particle is at rest initially when it is 5 m from O
and after 1 second when it is instantaneously at rest 4 m from O.

ii When $t = 0$, $a = -6$. The initial acceleration is $-6\ \mathrm{m\,s^{-2}}$.

Discussion point

How would you interpret the negative acceleration in the above example?

Exercise 16.1

1 In each of the following cases $t \geqslant 0$. The quantities are given in SI units, so distances are in metres and times in seconds.:

i find expressions for the velocity and acceleration at time t

ii use these expressions to find the initial position, velocity and acceleration

iii find the time and position when the velocity is zero.

a $s = 5t^2 - t + 3$

b $s = 3t - t^3$

c $s = t^4 - 4t - 6$

d $s = 4t^3 - 3t + 5$

e $s = 5 - 2t^2 + t$

2 A particle is projected in a straight line from a point O. After t seconds its displacement, s metres, from O is given by $s = 3t^2 - t^3$.

a Write expressions for the velocity and acceleration at time t.

b Find the times when the body is instantaneously at rest.

c What distance is travelled between these times?

d Find the velocity when $t = 4$ and interpret your result.

e Find the initial acceleration.

3 A ball is thrown upwards and its height, h metres, above ground after t seconds is given by $h = 1 + 4t - 5t^2$.

a From what height was the ball projected?

b Write an expression for the velocity of the ball at time t.

c When is the ball instantaneously at rest?

d What is the greatest height reached by the ball?

e After what length of time does the ball hit the ground?

f Sketch the graph of h against t.

g At what speed is the ball travelling when it hits the ground?

4 In the early stages of its motion the height of a rocket, h metres, is given by $h = \frac{1}{6}t^4$, where t seconds is the time after launch.

a Find expressions for the velocity and acceleration of the rocket at time t.

b After how long is the acceleration of the rocket $72\,\text{m}\,\text{s}^{-2}$?

c Find the height and velocity of the rocket at this time.

5 The velocity of a moving object at time t seconds is given by $v\,\text{m}\,\text{s}^{-1}$, where $v = 15t - 2t^2 - 25$.

a Find the times when the object is instantaneously at rest.

b Find the acceleration at these times.

c Find the velocity when the acceleration is zero.

d Sketch the graph of v against t.

Finding displacement from velocity and velocity from acceleration

In the previous section, you used the result $v = \frac{ds}{dt}$; in other words when s was given as an expression in t, you differentiated to find v. Therefore when v is given as an expression in t, integrating v gives an expression for s,

$$s = \int v \, dt.$$

Similarly, you can reverse the result $a = \frac{dv}{dt}$ to give

$$v = \int a \, dt.$$

Worked example

The acceleration is not constant.

A particle P moves in a straight line so that at time t seconds its acceleration is $(6t + 2)\,\text{m s}^{-2}$.

P passes through a point O at time $t = 0$ with a velocity of $3\,\text{m s}^{-1}$.

Find:

a the velocity of P in terms of t

b the distance of P from O when $t = 2$.

Solution

a
$$v = \int a \, dt$$
$$= \int (6t + 2) \, dt$$
$$= 3t^2 + 2t + c$$

When $t = 0$, $v = 3$
$\Rightarrow \quad c = 3.$

c represents the initial velocity.

Therefore $v = 3t^2 + 2t + 3$.

b
$$s = \int v \, dt$$
$$= \int (3t^2 + 2t + 3) \, dt$$
$$= t^3 + t^2 + 3t + k$$

k is the value of the displacement when $t = 0$.

When $t = 0$, $s = 0$
$\Rightarrow \quad k = 0$
$\Rightarrow \quad s = t^3 + t^2 + 3t$

When $t = 2$, $s = 8 + 4 + 6$
$= 18.$

When $t = 2$ the particle is 18 m from O.

Worked example

This tells you that the acceleration varies with time.

The acceleration of a particle $a\,\mathrm{m\,s^{-2}}$, at time t seconds is given by $a = 6 - 2t$.

When $t = 0$, the particle is at rest at a point 4 m from the origin O.

a Find expressions for the velocity and displacement in terms of t.

b Find when the particle is next at rest, and its displacement from O at that time.

Solution

a $v = \int \mathrm{a}\,\mathrm{d}t$

$= \int (6 - 2t)\,\mathrm{d}t$

$= 6t - t^2 + c$

When $t = 0$, $v = 0$ (given) $\Rightarrow$ $c = 0$

Therefore $v = 6t - t^2$.

$s = \int v\,\mathrm{d}t$

$= \int (6t - t^2)\,\mathrm{d}t$

$= 3t^2 - \frac{t^3}{3} + k$

When $t = 0$, $s = 4$ (given) $\Rightarrow$ $k = 4$

Therefore $s = 3t^2 - \frac{t^3}{3} + 4$.

b The particle is at rest when $v = 0$ $\Rightarrow$ $6t - t^2 = 0$

$\Rightarrow$ $t(6 - t) = 0$

$\Rightarrow$ $t = 0$ or $t = 6$

The particle is next at rest after 6 seconds.

When $t = 6$, $s = 3 \times 6^2 - \frac{6^3}{3} + 4$

$= 40$

The particle is 40 m from O after 6 seconds.

Worked example

A particle is projected along a straight line.

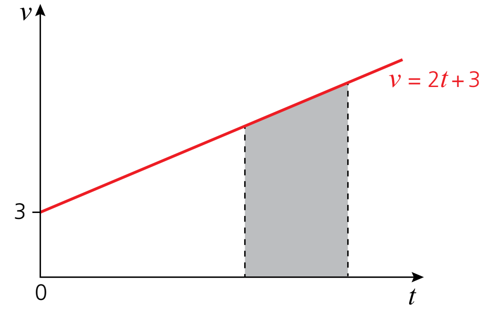
Its velocity, $v\,\mathrm{m\,s^{-1}}$, after t seconds is given by $v = 2t + 3$.

a Sketch the graph of v against t.

b Find the distance the particle moves in the third second.

Solution

a $v = 2t + 3$ is a straight line with gradient 2 that passes through (0, 3).



b **Method 1**

The graph shows that the velocity is always positive, so the velocity and speed are the same. The distance travelled is equal to the area under the graph.

The third second starts when $t = 2$ and finishes when $t = 3$.

Using the formula for the area of a trapezium,

$$\begin{aligned} \text{distance} &= \tfrac{1}{2}(7 + 9) \times 1 \\ &= 8\,\text{m}. \end{aligned}$$

Method 2

The area under a graph can also be found using integration.

$$\begin{aligned} \text{Distance} &= \int_a^b v \; \mathrm{d}t \\ &= \int_2^3 (2t + 3) \; \mathrm{d}t \\ &= [t^2 + 3t]_2^3 \\ &= [9 + 9] - [4 + 6] \\ &= 8\,\text{m} \end{aligned}$$

Discussion point

- Which method did you prefer to use in the previous example?
- Which method would you need to use if v was given by $v = 3t^2 + 2$?
  - Is acceleration constant in this case? How can you tell?
- Could you have used the constant acceleration (*suvat*) equations?
- Can you use calculus when acceleration is constant?

Exercise 16.2

1 Find expressions for the velocity, v, and displacement, s, at time t in each of the following cases:
  a $a = 2 - 6t$; when $t = 0$, $v = 1$ and $s = 0$
  b $a = 4t$; when $t = 0$, $v = 4$ and $s = 3$
  c $a = 12t^2 - 4$; when $t = 0$, $v = 2$ and $s = 1$
  d $a = 2$; when $t = 0$, $v = 2$ and $s = 4$
  e $a = 4 + t$; when $t = 0$, $v = 1$ and $s = 3$

2 A particle P sets off from the origin, O, with a velocity of $9\,\text{m s}^{-1}$ and moves along the x-axis.
At time t seconds, its acceleration is given by $a = (6t - 12)\,\text{m s}^{-2}$.
  a Find expressions for the velocity and displacement at time t.
  b Find the time when the particle returns to its starting point.

3 A particle P starts from rest at a fixed origin O when $t = 0$.
The acceleration $a\,\text{m s}^{-2}$ at time t seconds is given by $a = 6t - 6$.
  a Find the velocity of the particle after 1 second.
  b Find the time after leaving the origin when the particle is next instantaneously at rest, and the distance travelled to this point.

4 The speed, $v\,\text{m s}^{-1}$, of a car during braking is given by $v = 30 - 5t$, where t seconds is the time since the brakes were applied.
  a Sketch a graph of v against t.
  b How long does the car take to stop?
  c How far does it travel while braking?

5 A particle P moves in a straight line, starting from rest at the point O. t seconds after leaving O, the acceleration, $a\,\text{m s}^{-2}$, of P is given by $a = 4 + 12t$.
  a Find an expression for the velocity of the particle at time t.
  b Calculate the distance travelled by P in the third second.

6 The velocity $v\,\text{m s}^{-1}$, of a particle P at time t seconds is given by $v = t^3 - 4t^2 + 4t + 2$.
P moves in a straight line.
  a Find an expression for the acceleration, $a\,\text{m s}^{-2}$, in terms of t.
  b Find the times at which the acceleration is zero, and say what is happening between these times.
  c Find the distance travelled in the first three seconds.

Past–paper questions

1 A particle P moves in a straight line such that, t s after leaving a point O, its velocity v m s^{-1} is given by $v = 36t - 3t^2$ for $t \geqslant 0$.

(i) Find the value of t when the velocity of P stops increasing. [2]

(ii) Find the value of t when P comes to instantaneous rest. [2]

(iii) Find the distance of P from O when P is at instantaneous rest. [3]

(iv) Find the speed of P when P is again at O. [4]

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2 A particle travels in a straight line so that, t s after passing through a fixed point O, its velocity, v ms^{-1}, is given by $v = 3 + 6\sin 2t$.

(i) Find the velocity of the particle when $t = \frac{\pi}{4}$. [1]

(ii) Find the acceleration of the particle when $t = 2$. [3]

The particle first comes to instantaneous rest at the point P.

(iii) Find the distance OP. [5]

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3 A particle travels in a straight line so that, t s after passing through a fixed point O, its displacement s m from O is given by $s = \ln(t^2 + 1)$.

(i) Find the value of t when $s = 5$. [2]

(ii) Find the distance travelled by the particle during the third second. [2]

(iii) Show that, when $t = 2$, the velocity of the particle is 0.8 ms^{-1}. [2]

(iv) Find the acceleration of the particle when $t = 2$. [3]

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Learning outcomes

Now you should be able to:

★ apply differentiation and integration to kinematics problems that involve displacement, velocity and acceleration of a particle moving in a straight line with variable or constant acceleration, and the use of x–t and v–t graphs.

Key points

Quantity	Definition	S.I. unit	Unit symbol	Notation
Time	Measured from a fixed origin	second	s	t
Distance	Distance travelled in a given time	metre	m	x, y, s
Speed	Rate of change of distance	metre per second	$\mathrm{m\,s^{-1}}$	$v = \frac{dx}{dt}$ etc.
Displacement	Distance from a fixed origin	metre	m	x, y, s, h
Velocity	Rate of change of displacement	metre per second	$\mathrm{m\,s^{-1}}$	$v = \frac{ds}{dt}$
Acceleration	Rate of change of velocity	metre per second per second	$\mathrm{m\,s^{-2}}$	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ etc.

- ✔ For a **displacement–time** graph, the gradient is the velocity.
- ✔ For a **velocity–time** the gradient is the acceleration and the area under the graph is the displacement.
- ✔ For a **distance–time** graph the gradient is the speed.
- ✔ For general motion:
  - $s = \int v \, dt$ (Displacement is the area under a velocity–time graph.)
  - $v = \int a \, dt$