15 Integration

Growth is painful. Change is painful. But nothing is as painful as staying stuck where you do not belong.

N.R. Narayana Murthy (1946 –)

Discussion point

Mita is a long-distance runner. She carries a speed meter, which tells her what her speed is at various times during a race.

Time (hours)	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
Speed (metres per second)	4.4	4.4	4.4	4.6	5.0	5.2	0

What race do you think she was running?

How would you estimate her time?

Integration is the process of getting from a differential equation to the general solution.

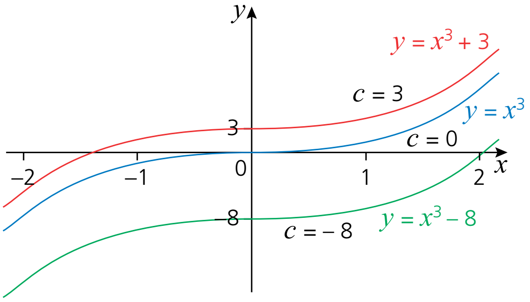
Integration involves using the rate of change of a quantity to find its total value at the end of an interval, for example using the speed of a runner to find the distance travelled at any time. The process is the reverse of differentiation.

Look at the **differential equation** $\frac{dy}{dx} = 3x^2$.

Since $\frac{dy}{dx} = 3x^2$ for x^3, $x^3 + 7$ and $x^3 - 3$, these expressions are all solutions of this equation.

The **general solution** of this differential equation is given as $y = x^3 + c$, where c is an **arbitrary constant** that can take any value (positive, negative or zero).

A solution containing an arbitrary constant gives a family of curves, as shown below. Each curve corresponds to a particular value of c.



Suppose that you are also given that the solution curve passes through the point (1, 4). Substituting these coordinates in $y = x^3 + c$ gives

$$4 = 1^3 + c \quad \Rightarrow \quad c = 3$$

So the equation of the curve is $y = x^3 + 3$. ← This is called the **particular solution.**

This example shows that if you know a point on a curve in the family, you can find the value of c and therefore the particular solution of a differential equation.

The rule for differentiation is usually given as

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}.$$

It can also be given as

$$y = x^{n+1} \quad \Rightarrow \quad \frac{dy}{dx} = (n+1)x^n$$

which is the same as

$$y = \frac{1}{n+1}x^{n+1} \quad \Rightarrow \quad \frac{dy}{dx} = x^n.$$

Note that if you are asked to integrate an expression f(x), this will mean integrate with respect to x unless otherwise stated.

Reversing this gives you the rule for integrating x^n. This is usually written using the integral symbol, $\int$.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad \text{for} \quad n \neq -1$$

The integral when $n = -1$ is a special case. If you try to apply the general rule, $n + 1$ is zero on the bottom line and so the expression you get is undefined. Instead you use the result that:

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$$

You will use this result later in the chapter.

Notice the use of dx on the left-hand side. This tells you that you are integrating with respect to x. So in this case you would read the left-hand side as 'The integral of x^n with respect to x'.

You may find it helpful to remember the rule as

- **add 1 to the power**
- **divide by the new power**
- **add a constant.**

Remember to include the arbitrary constant, c, until you have enough information to find a value for it.

Worked example

Integrate each of the following:

a x^6 b $5x^4$ c 7 d $4\sqrt{x}$

Solution

a $\frac{x^7}{7} + c$

b $5 \times \frac{x^5}{5} + c = x^5 + c$

c 7 can be thought of as $7x^0$ so applying the rule gives $7x + c$

d $4\sqrt{x} = 4x^{\frac{1}{2}}$ so applying the rule gives $\frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{8}{3}x^{\frac{3}{2}} + c$

Worked example

Given that $\frac{dy}{dx} = 6x^2 + 2x - 5$

a Find the general solution of this differential equation.

b Find the equation of the curve with this gradient function that passes through the point (1, 7).

Solution

a $y = 6 \times \frac{x^3}{3} + 2 \times \frac{x^2}{2} - 5x + c$ ← By integration

$= 2x^3 + x^2 - 5x + c$

b Since (1, 7) is a point on the graph

$7 = 2(1)^3 + 1^2 - 5 + c$

$\Rightarrow \quad c = 9$

$\Rightarrow \quad y = 2x^3 + x^2 - 5x + 9$

Worked example

Find f(x) given that f′(x) = $2x + 4$ and f(2) = –4.

Solution

$f'(x) = 2x + 4$

By integration → $f(x) = \frac{2x^2}{2} + 4x + c$

$= x^2 + 4x + c$

$f(2) = -4 \Rightarrow -4 = (2)^2 + 4(2) + c$

$\Rightarrow c = -16$

$\Rightarrow f(x) = x^2 + 4x - 16$

Worked example

A curve passes through (3, 5). The gradient of the curve is given by $\frac{dy}{dx} = x^2 - 4$.

a Find y in terms of x.

b Find the coordinates of any stationary points of the graph of y.

c Sketch the curve.

Solution

a $\frac{dy}{dx} = x^2 - 4 \Rightarrow y = \frac{x^3}{3} - 4x + c$

When $x = 3$, $5 = 9 - 12 + c$

$\Rightarrow c = 8$

So the equation of the curve is $y = \frac{x^3}{3} - 4x + 8$.

b $\frac{dy}{dx} = 0$ at all stationary points.

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x + 2)(x - 2) = 0$

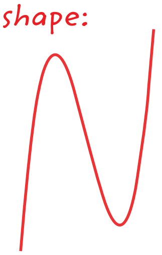
$\Rightarrow x = -2$ or $x = 2$

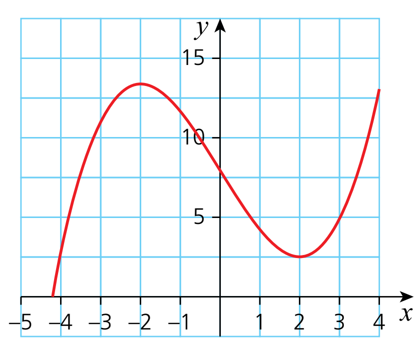
Substituting these values into the equation to find y

The stationary points are $(-2, 13\frac{1}{3})$ and $(2, 2\frac{2}{3})$.

c It crosses the y-axis at (0, 8).

The curve is a cubic with a positive x^3 term with two turning points, so it has this shape:





Worked example

Find $\int (x^3 - 2x^2)\, dx$.

Solution

$$\int (x^3 - 2x^2)\, dx = \frac{x^4}{4} - \frac{2x^3}{3} + c$$

Worked example

Find $\int (2x+1)(x-4)\, dx$

Solution

You need to multiply out the brackets before you can integrate.

$$\int (2x+1)(x-4)\, dx = \int (2x^2 - 7x - 4)\, dx$$

$$= \frac{2x^3}{3} - \frac{7x^2}{2} - 4x + c$$

Exercise 15.1

1 Find y in each of the following cases:

a $\frac{dy}{dx} = 4x + 2$

b $\frac{dy}{dx} = 6x^2 - 5x - 1$

c $\frac{dy}{dx} = 3 - 5x^3$

d $\frac{dy}{dx} = (x - 2)(3x + 2)$

2 Find $f(x)$ given that

a $f'(x) = 5x + 3$

b $f'(x) = x^4 + 2x^3 - x + 8$

c $f'(x) = (x - 4)(x^2 + 2)$

d $f'(x) = (x - 7)^2$

3 Find the following integrals:

a $\int 5\, dx$

b $\int 5x^3\, dx$

c $\int (2x - 3)\, dx$

d $\int (3x^3 - 4x + 3)\, dx$

4 Find the following integrals:

a $\int (3 - x)^2\, dx$

b $\int (2x + 1)(x - 3)\, dx$

c $\int (x + 1)^2\, dx$

d $\int (2x - 1)^2\, dx$

5 Find the equation of the curve $y = f(x)$ that passes through the specified point for each of the following gradient functions:

a $\frac{dy}{dx} = 2x - 3$; $(2, 4)$

b $\frac{dy}{dx} = 4 + 3x^3$ $(4, -2)$

c $\frac{dy}{dx} = 5x - 6$; $(-2, 4)$

d $f'(x) = x^2 + 1$; $(-3, -3)$

e $f'(x) = (x + 1)(x - 2)$; $(6, -2)$

f $f'(x) = (2x + 1)^2$; $(1, -1)$

Exercise 15.1 (cont)

6 Find the equation of the curve $y = \text{f}(x)$ that passes through the specified point for each of the following gradient functions:

a $\frac{\text{d}y}{\text{d}x} = 2\sqrt{x} - 1$; (1, 1)

b $\text{f}'(x) = x - \sqrt{x}$; (4, 2)

7 You are given that $\frac{\text{d}y}{\text{d}x} = 2x + 3$.

a Find $\int(2x + 3)\,\text{d}x$.

b Find the general solution of the differential equation.

c Find the equation of the curve with gradient function $\frac{\text{d}y}{\text{d}x}$ and that passes through (2, –1).

d Hence show that (–1, –13) lies on the curve.

8 The curve *C* passes through the point (3, 21) and its gradient at any point is given by $\frac{\text{d}y}{\text{d}x} = 3x^2 - 4x + 1$.

a Find the equation of the curve *C*.

b Show that the point (–2, –9) lies on the curve.

9 **a** Find $\int(4x - 1)\,\text{d}x$.

b Find the general solution of the differential equation $\frac{\text{d}y}{\text{d}x} = 4x - 1$.

c Find the particular solution that passes through the point (–1, 4).

d Does this curve pass above, below or through the point (2, 4)?

10 The curve $y = \text{f}(x)$ passes through the point (2, –4) and $\text{f}'(x) = 2 - 3x^2$. Find the value of f(–1).

11 A curve, C, has stationary points at the points where $x = 0$ and where $x = 2$.

a Explain why $\frac{\text{d}y}{\text{d}x} = x^2 - 2x$ is a possible expression for the gradient of C. Give a different possible expression for $\frac{\text{d}y}{\text{d}x}$.

b The curve passes through the point (3, 2).

Given that $\frac{\text{d}y}{\text{d}x}$ is $x^2 - 2x$, find the equation of C.

Definite integrals

So far, all the integrals you have met have been **indefinite integrals** such as $\int 3x^2\text{d}x$; the resulting expressions for *y* have all finished with '+ *c*'. You may or may not have been given additional information to enable you to find a value for *c*.

By contrast, a **definite integral** has two limits.

$$\int_1^3 3x^2\text{d}x$$

This is the upper limit. (3)

This is the lower limit. (1)

To find the value of a definite integral, you integrate it and substitute in the values of the limits. Then you subtract the value of the integral at the lower limit from the value of the integral at the upper limit.

Worked example

Find $\int_1^3 3x^2\,dx$.

Solution

Subtracting the value at $x = 1$ from the value at $x = 3$

$$\int 3x^2\,dx = x^3 + c$$

$$(3^3 + c) - (1^3 + c) = 26 \text{ so } \int_1^3 3x^2 dx = 26$$

Notice how the c is eliminated when you simplify this expression.

When evaluating definite integrals, it is common practice to omit the c and write

$$\int_1^3 3x^2\,dx = [x^3]_1^3 = [3^3] - [1^3] = 26.$$

The definite integral is defined as

$$\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a).$$

Worked example

'Evaluate' means 'find the numerical value of'.

Evaluate $\int_1^4 (x^2 + 3)\,dx$.

Solution

$$\int_1^4 (x^2 + 3)\,dx = \left[\frac{x^3}{3} + 3x\right]_1^4$$

$$= \left(\frac{4^3}{3} + 3\times 4\right) - \left(\frac{1^3}{3} + 3\times 1\right)$$

$$= 30$$

Worked example

Evaluate $\int_{-1}^3 (x + 1)(x - 3)\,dx$.

Solution

Notice how you need to expand $(x+1)(x+3)$ before integrating it.

$$\int_{-1}^3 (x + 1)(x - 3)\,dx = \int_{-1}^3 (x^2 - 2x - 3)\,dx.$$

$$= \left[\frac{x^3}{3} - x^2 - 3x\right]_{-1}^3$$

$$= \left(\frac{3^3}{3} - 3^2 - 3\times 3\right) - \left(\frac{(-1)^3}{3} - (-1)^2 - 3\times(-1)\right)$$

$$= -10\tfrac{2}{3}$$

Exercise 15.2 Evaluate the following definite integrals:

1 $\int_1^2 3x^2\,dx$

2 $\int_1^4 4x^3\,dx$

3 $\int_{-1}^1 6x^2\,dx$

4 $\int_1^5 4\,dx$

5 $\int_2^4 (x^2+1)\,dx$

6 $\int_{-2}^3 (2x+5)\,dx$

7 $\int_2^5 (4x^3-2x+1)\,dx$

8 $\int_5^6 (x^2-5)\,dx$

9 $\int_1^3 (x^2-3x+1)\,dx$

10 $\int_{-1}^2 (x^2+3)\,dx$

11 $\int_{-4}^{-1} (16-x^2)\,dx$

12 $\int_1^3 (x+1)(3-x)\,dx$

13 $\int_2^4 (3x(x+2))\,dx$

14 $\int_{-1}^1 (x+1)(x-1)\,dx$

15 $\int_{-1}^2 (x+4x^2)\,dx$

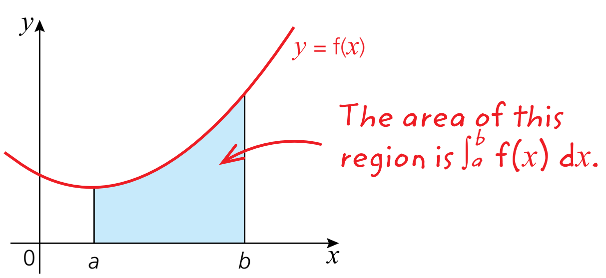
16 $\int_{-1}^1 x(x-1)(x+1)\,dx$

17 $\int_{-1}^3 (x^3+2)\,dx$

18 $\int_{-3}^1 (9-x^2)\,dx$

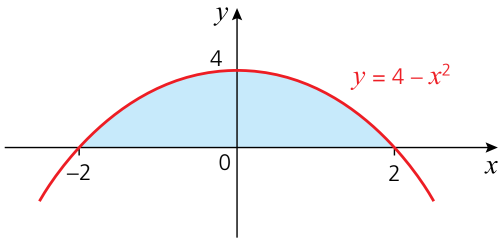
Finding the area between a graph and the x-axis

The area under the curve, $y = f(x)$ between $x = a$ and $x = b$, the shaded region in the graph below, is given by a definite integral: $\int_a^b f(x)\,dx$.



Worked example

Find the area of the shaded region under the curve $y = 4 - x^2$.



Solution

$$\text{Area} = \int_{-2}^{2} (4 - x^2)\ \mathrm{d}x = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

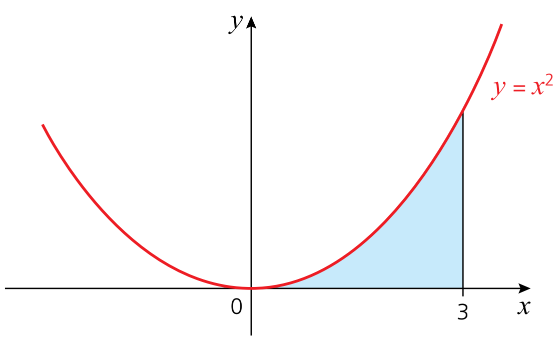
$$= \left[4 \times 2 - \frac{2^3}{3}\right] - \left[4 \times (-2) - \frac{(-2)^3}{3}\right]$$

$$= 10\frac{2}{3}\ \text{units}^2$$

Exercise 15.3

Find the area of each of the shaded regions:

1



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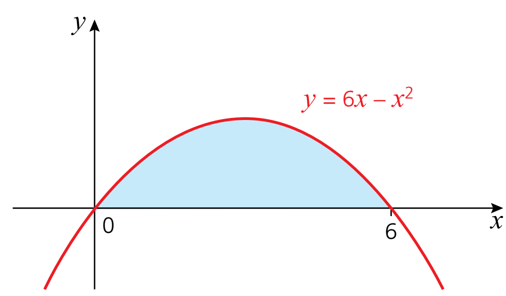

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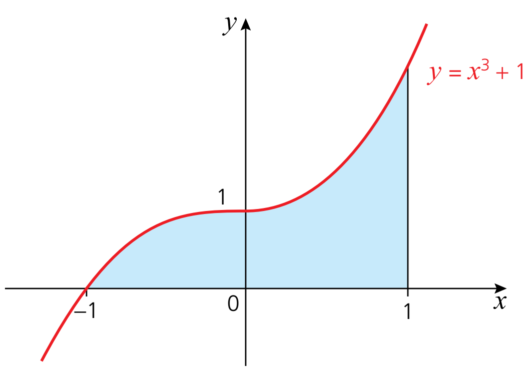

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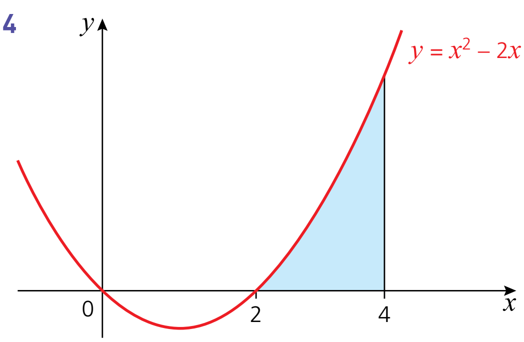

Exercise 15.3 (cont)

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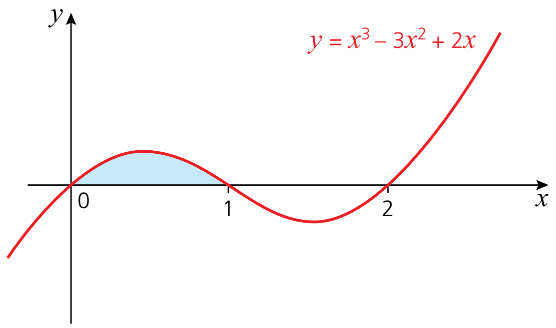

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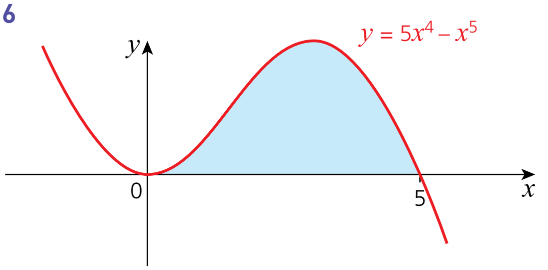

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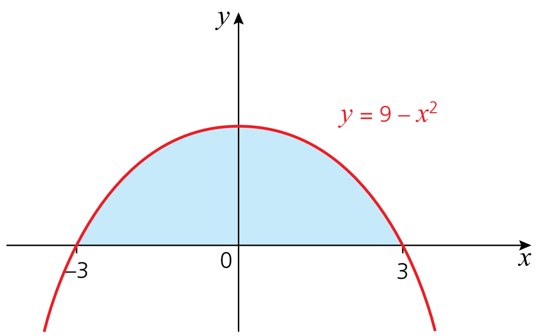

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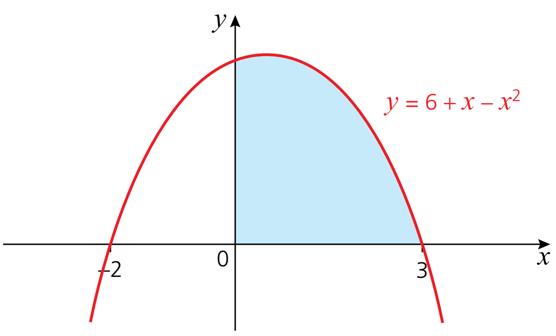

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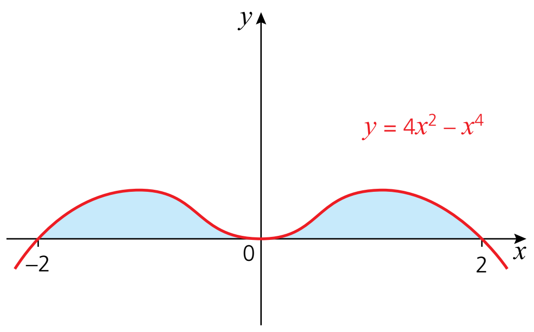

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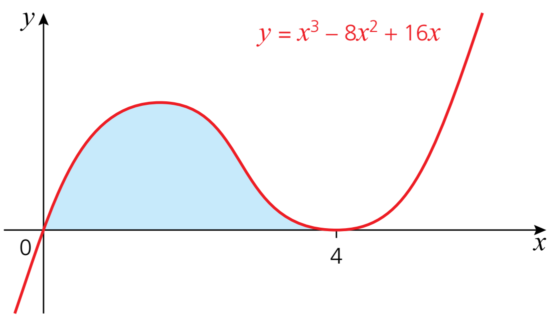
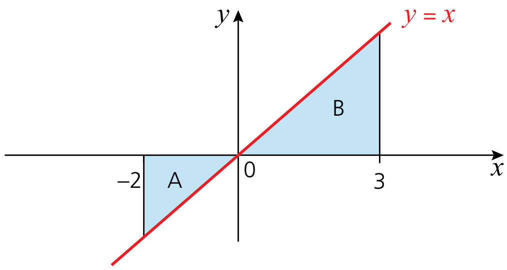


So far all the areas you have found have been above the x-axis. The next example involves a region that is below the x-axis.

Worked example

The diagram shows the line $y = x$ and two regions marked A and B.



a Calculate the areas of A and B using the formula for the area of a triangle.

b Evaluate $\int_{-2}^{0} x\,\mathrm{d}x$ and $\int_{0}^{3} x\,\mathrm{d}x$. What do you notice?

c Evaluate $\int_{-2}^{3} x\,\mathrm{d}x$. What do you notice?

Solution

a Area of A $= \frac{1}{2} \times 2 \times 2 = 2$ square units.

Area of B $= \frac{1}{2} \times 3 \times 3 = 4.5$ square units.

b $$\int_{-2}^{0} x\,\mathrm{d}x = \left(\frac{x^2}{2}\right)_{-2}^{0}$$
$$= 0 - (2)$$
$$= -2$$

$$\int_{0}^{3} x\,\mathrm{d}x = \left(\frac{x^2}{2}\right)_{0}^{3}$$
$$= 4.5 - 0$$
$$= 4.5$$

The areas have the same numerical values as the integral but when the area is below the x-axis, the integral is negative.

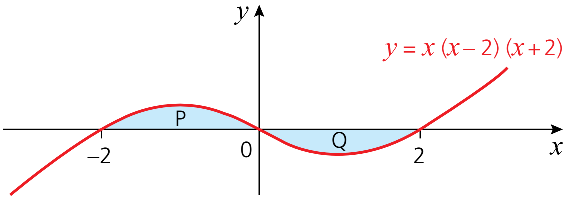
c $$\int_{-2}^{3} x\,\mathrm{d}x = \left(\frac{x^2}{2}\right)_{-2}^{3} = 4.5 - (2)$$
$$= 4.5 - (2)$$
$$= 2.5$$

The areas above and below the x-axis have cancelled each other out.

This example shows you how using integration gives a negative answer for the area of a region below the x-axis. In some contexts this will make sense and in others it won't, so you always have to be careful.

Worked example

The curve $y = x(x-2)(x+2)$ is drawn on the axes.



a Use integration to find the areas of each of the shaded regions P and Q.

b Evaluate $\int_{-2}^{2} x(x-2)(x+2)\,dx$.

c What do you notice?

Solution

a Area of P: $\int_{-2}^{0} x(x-2)(x+2)\,dx = \int_{-2}^{0} (x^3-4x)\,dx$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}$$

$$= 0 - \left[\frac{(-2)^4}{4} - 2\times(-2)^2\right]$$

$$= 4$$

So P has an area of 4 units^2.

Area of Q: $\int_{0}^{2} x(x-2)(x+2)\,dx = \int_{0}^{2} (x^3-4x)\,dx$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= \left[\frac{2^4}{4} - 2\times 2^2\right] - 0$$

$$= -4$$

So Q also has an area of 4 units^2.

The areas of P and Q are the same since the curve has rotational symmetry about the origin.

b $\int_{-2}^{2} x(x-2)(x+2)\,dx = \int_{-2}^{2} (x^3-4x)\,dx$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{2}$$

$$= \left[\frac{2^4}{4} - 2\times 2^2\right] - \left[\frac{(-2)^4}{4} - 2\times(-2)^2\right]$$

$$= 0$$

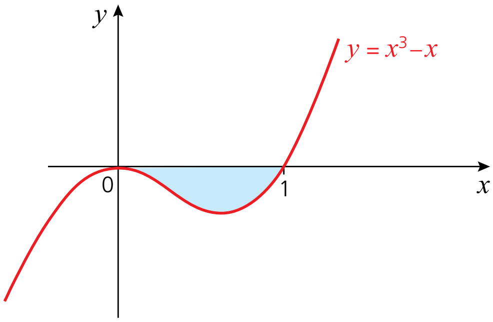
Always draw a sketch graph when you are going to calculate areas. This will avoid any cancelling out of areas above and below the x-axis.

c The areas of P and Q have ‘cancelled out’.

Exercise 15.4

1 The sketch shows the curve $y = x^3 - x$.

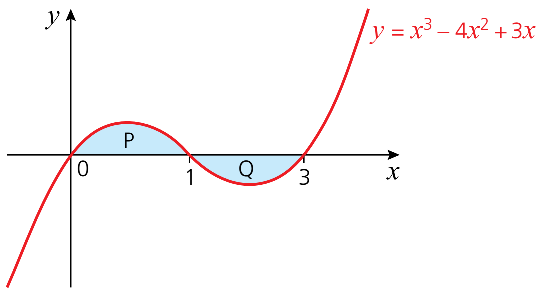
Calculate the area of the shaded region.



2 The sketch shows the curve $y = x^3 - 4x^2 + 3x$.

a Calculate the area of each shaded region.

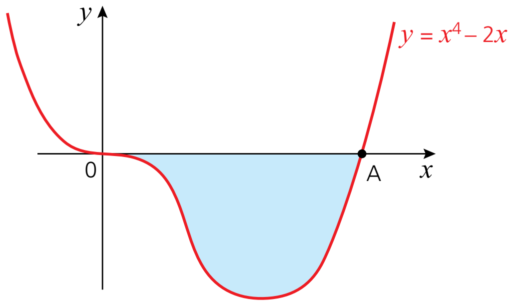
b State the area enclosed between the curve and the x-axis.



3 The sketch shows the curve $y = x^4 - 2x$.

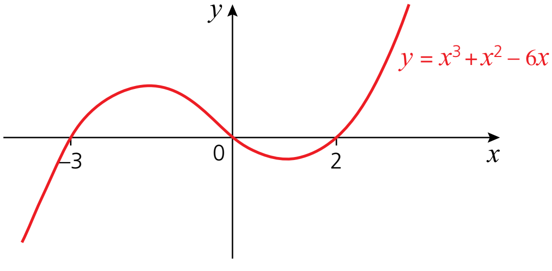
a Find the coordinates of the point A.

b Calculate the area of the shaded region.



4 The sketch shows the curve $y = x^3 + x^2 - 6x$.

Work out the area between the curve and the x-axis.



Exercise 15.4 (cont)

5 **a** Sketch the curve $y = x^2$ for $-3 < x < 3$.
b Shade the area bounded by the curve, the lines $x = -1$ and $x = 2$ and the x-axis.
c Find, by integration, the area of the region you have shaded.

6 **a** Sketch the curve $y = x^2 - 2x$ for $-1 < x < 3$.
b For what values of x does the curve lie below the x-axis?
c Find the area between the curve and the x-axis.

7 **a** Sketch the curve $y = x^3$ for $-3 < x < 3$.
b Shade the area between the curve, the x-axis and the line $x = 2$.
c Find, by integration, the area of the region you have shaded.
d Without any further calculation, state, with reasons, the value of $\int_{-2}^{2} x^3\,\mathrm{d}x$.

8 **a** Shade, on a suitable sketch, the region with an area given by $\int_{-1}^{2} (x^2 + 1)\,\mathrm{d}x$.
b Evaluate this integral.

9 **a** Evaluate $\int_{1}^{4} (2x + 1)\,\mathrm{d}x$.
b Interpret this integral on a sketch graph.

Integrating other functions of x

As with differentiation, there are a number of special cases when integrating. The proofs are beyond the scope of this book, but you will need to know and be able to use these results.

Integrating $\frac{1}{x}$ (i.e. x^{-1}) does not follow the general rule since that would give $\frac{x^0}{0}$ which is undefined.

Differentiation ⇒	Basic integral ⇒	Generalised integral
$y = x^n \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$, for $n \neq -1$	$\int (ax+b)^n\,\mathrm{d}x = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$, for $n \neq -1$
$y = \sin x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$	$\int \cos x\,\mathrm{d}x = \sin x + c$	$\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$
$y = \cos x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$	$\int \sin x\,\mathrm{d}x = -\cos x + c$	$\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c$
$y = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$	$\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + c$	$\int \mathrm{e}^{(ax+b)}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{(ax+b)} + c$
$y = \ln x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$	$\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x\rvert + c$	$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln\lvert ax+b\rvert + c$

Worked example

Find the following indefinite integrals:

a $\int \frac{1}{2x-3}\,dx$ **b** $\int (2x-3)^4\,dx$ **c** $\int (2x-3)^{\frac{1}{2}}\,dx$

d $\int e^{2x-3}dx$ **e** $\int \sin(2x-3)\,dx$ **f** $\int \cos(2x-3)\,dx$

Solution

a Using $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+c$

gives $\int \frac{1}{2x-3}\,dx = \frac{1}{2}\ln|2x-3|+c$

b Using $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c$

gives $\int (2x-3)^4\,dx = \frac{1}{2}\frac{(2x-3)^5}{5}+c = \frac{(2x-3)^5}{10}+c$

c Using $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c$

gives $\int (2x-3)^{\frac{1}{2}}\,dx = \frac{1}{2}\frac{(2x-3)^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{1}{3}(2x-3)^{\frac{3}{2}}+c$

d Using $\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b}+c$

gives $\int e^{2x-3}dx = \frac{1}{2}e^{2x-3}+c$

e Using $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$

gives $\int \sin(2x-3)\,dx = -\frac{1}{2}\cos(2x-3)+c$

f Using $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

gives $\int \cos(2x-3)\,dx = \frac{1}{2}\sin(2x-3)+c$

When integrating trigonometric functions, the angles must be in radians

Worked example

Evaluate the following definite integrals:

a $\int_2^3 \frac{1}{2x+1}\,dx$ **b** $\int_2^3 (2x+1)^4\,dx$ **c** $\int_2^3 e^{2x+1}\,dx$

d $\int_0^1 (2x+1)^{\frac{1}{2}}\,dx$ **e** $\int_0^1 (2x+1)^{-2}\,dx$ **f** $\int_0^{\frac{\pi}{3}} \sin\left(2x+\frac{\pi}{6}\right)dx$

g $\int_0^{\frac{\pi}{3}} \cos\left(2x+\frac{\pi}{6}\right)dx$

Solution

Using $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+c$

a $\int_2^3 \frac{1}{2x+1}\,dx = \left[\frac{1}{2}\ln|2x+1|\right]_2^3$

$= \frac{1}{2}\ln 7 - \frac{1}{2}\ln 5$

$= \frac{1}{2}(\ln 7 - \ln 5)$

$= \frac{1}{2}\ln\frac{7}{5}$

Using $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c$

b $\int_2^3 (2x+1)^4\,dx = \left[\frac{1}{2}\frac{(2x+1)^5}{5}\right]_2^3$

$= \frac{1}{2}\left[\frac{7^5}{5} - \frac{5^5}{5}\right]$

$= 1368.2$

Using $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}+c$

c $\int_2^3 e^{2x+1}\,dx = \left[\frac{1}{2}e^{2x+1}\right]_2^3$

$= \frac{1}{2}(e^7 - e^5)$

d $\int_0^1 (2x+1)^{\frac{1}{2}}\,dx = \left[\frac{1}{2}\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$

$= \frac{1}{3}\left[3^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= 1.40$ (3s.f.)

e $\int_0^1 (2x+1)^{-2}\,dx = \left[\frac{1}{2}\frac{(2x+1)^{-1}}{-1}\right]_0^1$

$= -\frac{1}{2}[3^{-1} - 1^{-1}]$

$= \frac{1}{3}$

Using $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$

f $\int_0^{\frac{\pi}{3}} \sin\left(2x+\frac{\pi}{6}\right)dx = \left[-\frac{1}{2}\cos\left(2x+\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$

$= -\frac{1}{2}\cos\frac{5\pi}{6} + \frac{1}{2}\cos\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}$

Using $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

g $\int_0^{\frac{\pi}{3}} \cos\left(2x+\frac{\pi}{6}\right)dx = \left[\frac{1}{2}\sin\left(2x+\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\sin\frac{5\pi}{6} - \frac{1}{2}\sin\frac{\pi}{6}$

$= 0$

Exercise 15.5

1 Find the following indefinite integrals:

a $\int \frac{1}{3x+1}dx$ **b** $\int (3x+1)^4\,dx$ **c** $\int e^{3x+1}dx$

d $\int \sin(3x+1)dx$ **e** $\int \cos(3x+1)dx$ **f** $\int \frac{3}{x-3}dx$

g $\int (2x-1)^3\,dx$ **h** $\int 4e^{2x-3}dx$ **i** $\int 3\sin(3x)dx$

j $\int 4\cos\left(\frac{x}{2}\right)dx$ **k** $\int (x-2)^{\frac{3}{2}}\,dx$ **l** $\int (2x-1)^{\frac{3}{2}}\,dx$

2 Evaluate the following definite integrals:

a $\int_2^4 \frac{1}{3x+1}dx$ **b** $\int_2^4 (3x+1)^4\,dx$ **c** $\int_2^4 e^{3x+1}\,dx$

d $\int_0^{\frac{\pi}{3}} \sin\left(3x+\frac{\pi}{3}\right)dx$ **e** $\int_0^{\frac{\pi}{3}} \cos\left(3x+\frac{\pi}{3}\right)dx$ **f** $\int_4^8 \frac{4}{x-2}dx$

g $\int_{-1}^3 (2x+3)^4\,dx$ **h** $\int_0^2 10e^{-2x}\,dx$ **i** $\int_0^{\frac{\pi}{2}} \sin\left(2x-\frac{\pi}{4}\right)dx$

j $\int_0^{\frac{\pi}{2}} \cos\left(2x-\frac{\pi}{4}\right)dx$

Past–paper questions

1 **(a)** A curve is such that $\frac{dy}{dx} = ae^{1-x} - 3x^2$, where a is a constant. At the point (1, 4), the gradient of the curve is 2.

(i) Find the value of a. [1]

(ii) Find the equation of the curve. [5]

(b) **(i)** Find $\int (7x+8)^{\frac{1}{3}}dx$. [2]

(ii) Hence evaluate $\int_0^8 (7x+8)^{\frac{1}{3}}dx$. [2]

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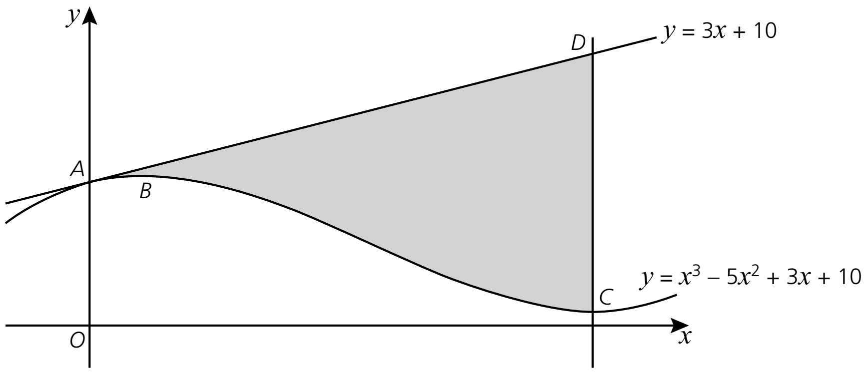
2



The diagram shows parts of the line $y = 3x + 10$ and the curve $y = x^3 - 5x^2 + 3x + 10$. The line and the curve both pass through the point A on the y-axis. The curve has a maximum at the point B and a minimum at the point C. The line through C, parallel to the y-axis, intersects the line $y = 3x + 10$ at the point D.

(i) Show that the line AD is a tangent to the curve at A. [2]

(ii) Find the x-coordinate of B and of C. [3]

(iii) Find the area of the shaded region $ABCD$, showing all your working. [5]

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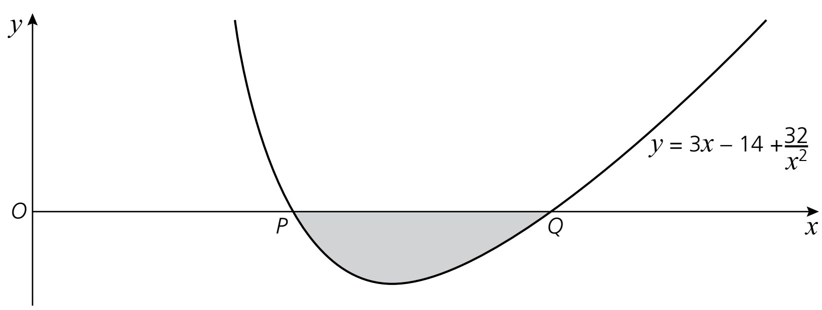
3 The diagram below shows part of the curve $y = 3x - 14 + \dfrac{32}{x^2}$ cutting the x-axis at the points P and Q.



(iii) State the x-coordinates of P and Q. [1]

(iv) Find $\int \left(3x - 14 + \dfrac{32}{x^2}\right) \mathrm{d}x$ and hence determine the area of the shaded region. [4]

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(Part question: parts (i) and (ii) omitted)
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(Part question: parts (i) and (ii) omitted)

Learning outcomes

Now you should be able to:

- ★ understand integration as the reverse process of differentiation
- ★ evaluate definite integrals and apply integration to the evaluation of plane areas
- ★ integrate sums of terms in powers of x including $\frac{1}{x}$ and $\frac{1}{ax+b}$
- ★ integrate functions of the form $(ax + b)^n$ for any rational n, $\sin(ax + b)$, $\cos(ax + b)$, e^{ax+b}

Key points

- ✔ $\frac{dy}{dx} = x^n \Rightarrow y = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$

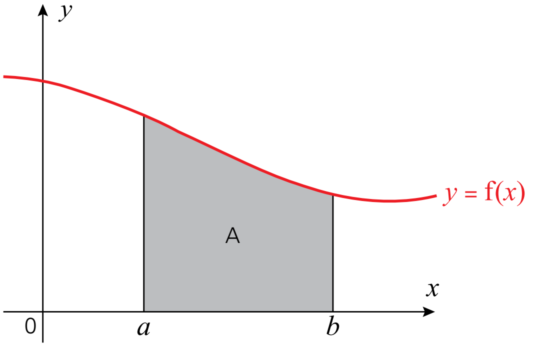
  This is an indefinite integral.

- ✔ $\int_a^b x^n \, dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$ for $n \neq -1$

  This is a definite integral.

- ✔ The area of a region between a curve $y = f(x)$ and the x-axis is given by $\int_a^b y \, dx$.

  Area of A $= \int_a^b y \, dx = \int_a^b f(x) \, dx$



- ✔ Areas below the x-axis give rise to negative values for the integral.
- ✔ Integrals of other functions where c is a constant:

Function $y = \mathbf{f}(x)$	**Integral** $\int y \, \mathbf{d}x$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert + c$
$(ax+b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
e^{ax+b}	$\frac{1}{a}e^{ax+b} + c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$