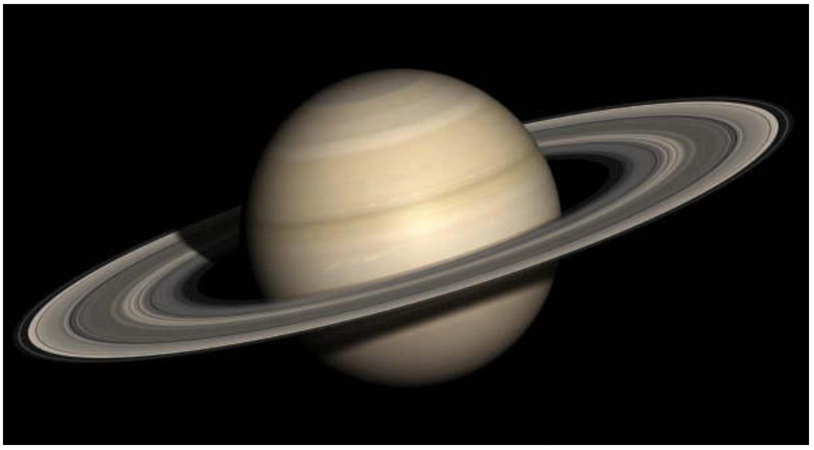
14 Differentiation

If I have seen further than others, it is by standing upon the shoulders of giants.

Isaac Newton (1642–1727)

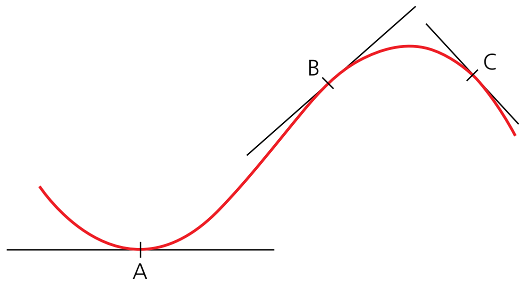


Discussion point

Look at the planet Saturn in the image above. What connection did Newton make between an apple and the motion of the planets?

In Newton's early years, mathematics was not advanced enough to enable people to calculate the orbits of the planets round the sun. In order to address this, Newton invented calculus, the branch of mathematics that you will learn about in this chapter.

The gradient function



The curve in the diagram has a zero gradient at A, a positive gradient at B and a negative gradient at C.

Although you can calculate the gradient of a curve at a given point by drawing a tangent at that point and using two points on the tangent to calculate its gradient, this process is time-consuming and the results depend on the accuracy of your drawing and measuring. If you know the equation of the curve, you can use **differentiation** to calculate the gradient.

Worked example

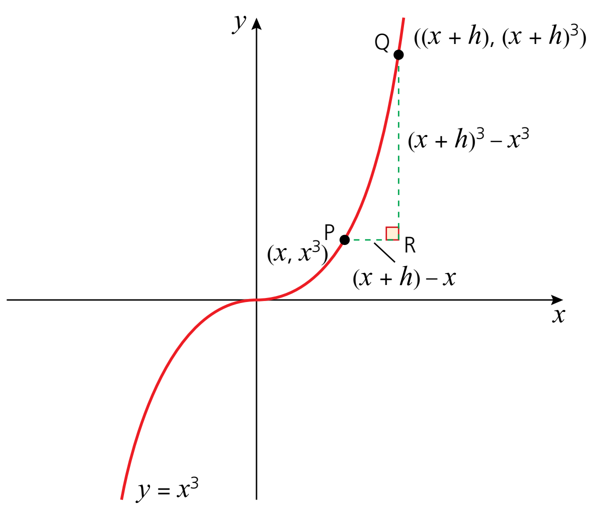
Work out the gradient of the curve $y = x^3$ at the general point (x, y).

Solution

Let P have the general value x as its x-coordinate, so P is the point (x, x^3).

Let the x-coordinate of Q be $(x + h)$ so Q is the point $((x + h), (x + h)^3)$.

Since it is on the curve $y = x^3$



The gradient of the chord PQ is given by

$$\frac{QR}{PR} = \frac{(x+h)^3 - x^3}{(x+h) - x}$$
$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \frac{3x^2h + 3xh^2 + h^3}{h}$$
$$= \frac{h(3x^2 + 3xh + h^2)}{h}$$
$$= 3x^2 + 3xh + h^2$$

As Q gets closer to P, h takes smaller and smaller values and the gradient approaches the value of $3x^2$, which is the gradient of the tangent at P.

The gradient of the curve $y = x^3$ at the point (x, y) is equal to $3x^2$.

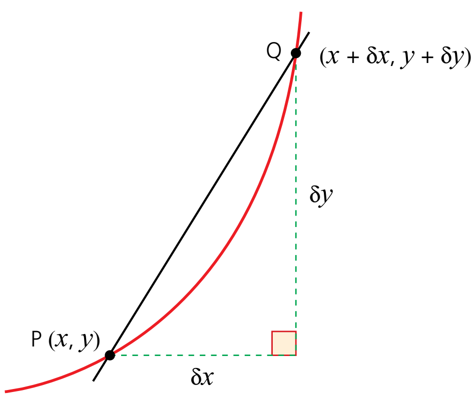
The gradient function is the gradient of the curve at the general point (x, y).

If the equation of the curve is written as $y = f(x)$, then the **gradient function** is written as $f'(x)$. Using this notation, the result in the previous example can be written as

$$f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2.$$

In the previous example, h was used to denote the difference between the x-coordinates of the points P and Q, where Q is close to P.

h is sometimes replaced by δx. The Greek letter δ (delta) is shorthand for 'a small change in' and so δx represents a small change in x, δy a small change in y and so on.



In the diagram the gradient of the chord PQ is $\frac{\delta y}{\delta x}$.

In the limit as δx tends towards 0, δx and δy both become infinitesimally small and the value obtained for $\frac{\delta y}{\delta x}$ approaches the gradient of the tangent at P.

$$\lim \frac{\delta y}{\delta x} \text{ is written as } \frac{dy}{dx}.$$

Using this notation, you have a rule for differentiation.

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

The gradient function, $\frac{dy}{dx}$, is sometimes called the **derivative** of y with respect to x. When you find it you have **differentiated** y with respect to x.

If the curve is written as $y = f(x)$, then the derivative is $f'(x)$.

If you are asked to differentiate a relationship in the form $y = f(x)$ in this book, this means differentiate with respect to x unless otherwise stated.

Note

There is nothing special about the letters x, y or f. If, for example, your curve represents time, t, on the horizontal axis and velocity, v, on the vertical axis, then the relationship could be referred to as $v = g(t)$. In this case v is a function of t and the gradient function is given by $\frac{dv}{dt} = g'(t)$.

The differentiation rule

Although it is possible to find the gradient from first principles which establishes a formal basis for differentiation, in practice you will use the differentiation rule introduced above;

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}.$$

You can also use this rule to differentiate (find the gradient of) equations that represent straight lines. For example, the gradient of the line $y = x$ is the same as $y = x^1$, so using the rule for differentiation, $\frac{dy}{dx} = 1 \times x^0 = 1$.

Lines of the form $y = c$ are parallel to the x-axis.

The gradient of the line $y = c$ where c is a constant is 0. For example, $y = 4$ is the same as $y = 4x^0$ so using the rule for differentiation, $\frac{dy}{dx} = 4 \times 0 \times x^{-1} = 0$. In general, differentiating any constant gives zero.

The rule can be extended further to include functions of the type $y = kx^n$ for any constant k, to give

This result is true for all powers of x, positive, negative and fractional.

$$y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = nkx^{n-1}.$$

You may find it helpful to remember the rule as

multiply by the power of x and reduce the power by 1.

Worked example

For each function, find the gradient function.

a $y = x^7$ **b** $u = 4x^3$ **c** $v = 5t^2$

d $y = 4x^{-3}$ **e** $P = 4\sqrt{t}$ **f** $y = \dfrac{4x^3 - 5}{x^2}$

Solution

a $y = x^7$

$$\Rightarrow \frac{dy}{dx} = 7x^6$$

b $u = 4x^3$

$$\Rightarrow \frac{du}{dx} = 4 \times 3x^2 = 12x^2$$

c $v = 5t^2$

$$\Rightarrow \frac{dv}{dt} = 5 \times 2t = 10t$$

d $y = 4x^{-3}$

$$\Rightarrow \frac{dy}{dx} = 4 \times (-3)x^{-3-1} = -12x^{-4}$$

e $P = 4t^{\frac{1}{2}}$

Using $\sqrt{t} = t^{\frac{1}{2}}$

$$\Rightarrow \frac{dP}{dt} = 4 \times \frac{1}{2}t^{\frac{1}{2}-1}$$

$$= \frac{2}{t^{\frac{1}{2}}}$$

$$= \frac{2}{\sqrt{t}}$$

f $y = \dfrac{4x^3 - 5}{x^2}$

$$\Rightarrow y = \frac{4x^3 - 5}{x^2} \Rightarrow y = \frac{4x^3}{x^2} - \frac{5}{x^2}$$

$$\Rightarrow y = 4x - 5x^{-2}$$

$$\frac{dy}{dx} = 4 + 10x^{-3}$$

$$= 4 + \frac{10}{x^3}$$

Sums and differences of functions

Many of the functions you will meet are sums or differences of simpler functions. For example, the function $(4x^3 + 3x)$ is the sum of the functions $4x^3$ and $3x$. To differentiate these functions, differentiate each part separately and then add the results together.

Worked example

Differentiate $y = 4x^3 + 3x$.

Solution

$$\frac{dy}{dx} = 12x^2 + 3$$

This example illustrates the general result that

$$y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x).$$

Worked example

Given that $y = 2x^3 - 3x + 4$, find

a $\frac{dy}{dx}$

b the gradient of the curve at the point (2, 14).

Solution

a $\frac{dy}{dx} = 6x^2 - 3$

b At (2, 14), $x = 2$.

$$\frac{dy}{dx} = 6 \times (2)^2 - 3 = 21$$

Substituting $x = 2$ in the expression for $\frac{dy}{dx}$

Exercise 14.1

Differentiate the following functions using the rules

$$y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = nkx^{n-1}$$

and $$y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x).$$

1 **a** $y = x^4$ **b** $y = 2x^3$ **c** $y = 5$ **d** $y = 10x$

2 **a** $y = x^{\frac{1}{2}}$ **b** $y = 5\sqrt{x}$ **c** $P = 7t^{\frac{3}{2}}$ **d** $y = \frac{1}{5}x^{\frac{5}{2}}$

3 **a** $y = 2x^5 + 4x^2$ **b** $y = 3x^4 + 8x$ **c** $y = x^3 + 4$ **d** $y = x - 5x^3$

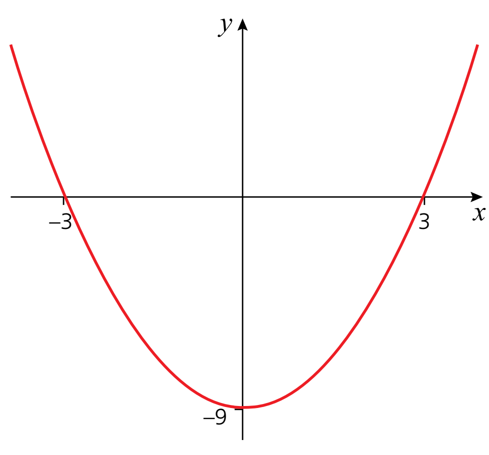
4 **a** $f(x) = \frac{1}{x^2}$ **b** $f(x) = \frac{6}{x^3}$

c $f(x) = 4\sqrt{x} - \frac{8}{\sqrt{x}}$ **d** $f(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

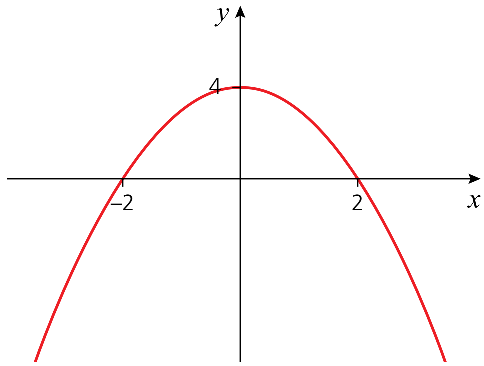
5 **a** $y = x(x - 1)$ **b** $y = (x + 1)(2x - 3)$

c $y = \frac{x^3 + 5x}{x^2}$ **d** $y = x\sqrt{x}$

6 Find the gradient of the curve $y = x^2 - 9$ at the points of intersection with the x- and y-axes.



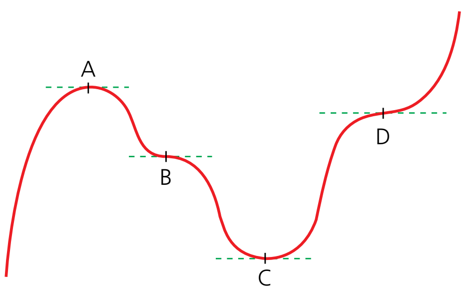
7 **a** Copy the curve of $y = 4 - x^2$ and draw the graph of $y = x - 2$ on the same axes.



b Find the coordinates of the points where the two graphs intersect.
c Find the gradient of the curve at the points of intersection.

Stationary points

A **stationary point** is a point on a curve where the gradient is zero. This means that the tangents to the curve at these points are horizontal. The diagram shows a curve with four stationary points: A, B, C and D.



The points A and C are **turning points** of the curve because as the curve passes through these points, it changes direction completely: at A the gradient changes from positive to negative and at C from negative to positive. A is called a **maximum** turning point, and C is a **minimum** turning point.

At B the curve does not turn: the gradient is negative both to the left and to the right of this point. B is a **stationary point of inflection**.

Discussion point

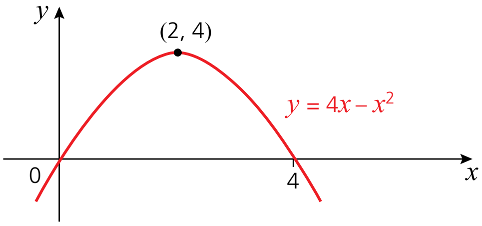
What can you say about the gradient to the left and right of D?

Note

Points where a curve 'twists' but doesn't have a zero gradient are also called points of inflection. However, in this section you will look only at *stationary* points of inflection. The tangent at a point of inflection both touches and intersects the curve.

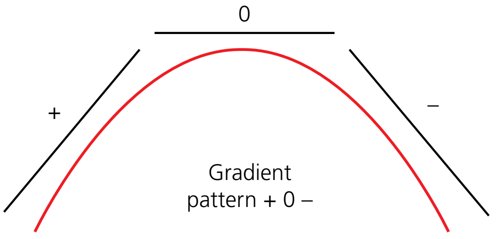
Maximum and minimum points

The graph shows the curve of $y = 4x - x^2$.



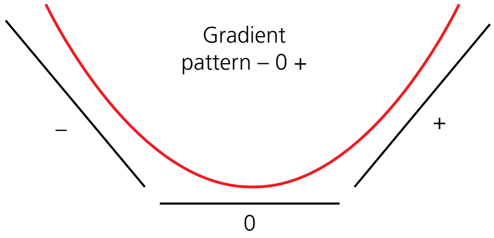
- The curve has a *maximum* point at (2, 4).
- The gradient $\frac{dy}{dx}$ at the maximum point is zero.
- The gradient is positive to the left of the maximum and negative to the right of it.

This is true for any maximum point as shown below.



For any minimum turning point, the gradient

- is zero at that point
- goes from negative to zero to positive.



Once you can find the position of any stationary points, and what type of points they are, you can use this information to help you sketch graphs.

Worked example

a For the curve $y = x^3 - 12x + 3$

- **i** find $\dfrac{dy}{dx}$ and the values of x for which $\dfrac{dy}{dx} = 0$
- **ii** classify the points on the curve with these values of x
- **iii** find the corresponding values of y
- **iv** sketch the curve.

b Why can you be confident about continuing the sketch of the curve beyond the x-values of the turning points?

c You did not need to find the coordinates of the points where the curve crosses the x-axis before sketching the graph. Why was this and under what circumstances would you find these points?

Solution

a i $\dfrac{dy}{dx} = 3x^2 - 12$

When $\dfrac{dy}{dx} = 0$, $3x^2 - 12 = 0$

$\Rightarrow \quad 3(x^2 - 4) = 0$

$\Rightarrow \quad 3(x + 2)(x - 2) = 0$

$\Rightarrow \quad x = -2$ or $x = 2$

Looking at the gradient pattern around $x = -2$

ii When $x = -3$, $\dfrac{dy}{dx} = 3(-3)^2 - 12 = 15$.

When $x = -1$ $\dfrac{dy}{dx} = 3(-1)^2 - 12 = -9$.

The gradient pattern is + 0 –

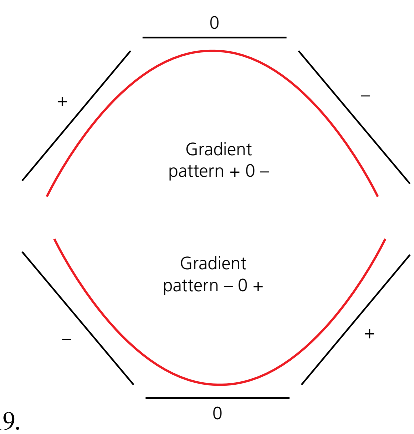
$\Rightarrow$ maximum turning point at $x = -2$.

When $x = 1$, $\dfrac{dy}{dx} = 3(1)^2 - 12 = -9$

When $x = 3$ $\dfrac{dy}{dx} = 3(3)^2 - 12 = 15$

The gradient pattern is – 0 +

$\Rightarrow$ minimum turning point at $x = +2$

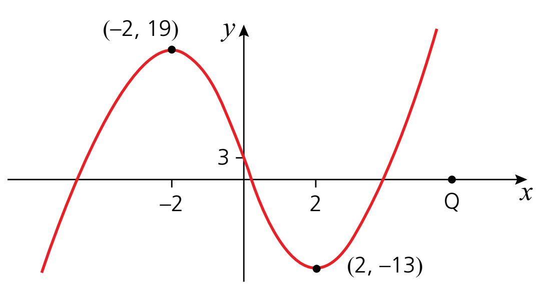


(–2, 19) is a maximum and (2, –13) a minimum.

iii When $x = -2$, $y = (-2)^3 - 12(-2) + 3 = 19$.
When $x = +2$, $y = (2)^3 - 12(2) + 3 = -13$.

The value of y when $x = 0$ tells you where the curve crosses the y-axis.

iv When $x = 0$, $y = (0)^3 - 12(0) + 3 = 3$.



b A cubic has at most 2 turning points and they have both been found. So the parts of the curve beyond them (to the left and to the right) just get steeper and steeper.

c The sketch is showing the shape of the curve and this is not affected by where it crosses the axes. However, you can see from the equation that it crosses the y-axis at (0, 3) and it is good practice to mark this in.

Worked example

Find all the turning points on the graph of $y = t^4 - 2t^3 + t^2 - 2$ and then sketch the curve.

Solution

$$\frac{dy}{dt} = 4t^3 - 6t^2 + 2t$$

$$\begin{aligned}\frac{dy}{dt} = 0 \quad &\Rightarrow \quad 4t^3 - 6t^2 + 2t = 0\\ &\Rightarrow \quad 2t(2t^2 - 3t + 1) = 0\\ &\Rightarrow \quad 2t(2t - 1)(t - 1) = 0\\ &\Rightarrow \quad t = 0 \text{ or } t = 0.5 \text{ or } t = 1\end{aligned}$$

Turning points occur when $\frac{dy}{dt} = 0$.

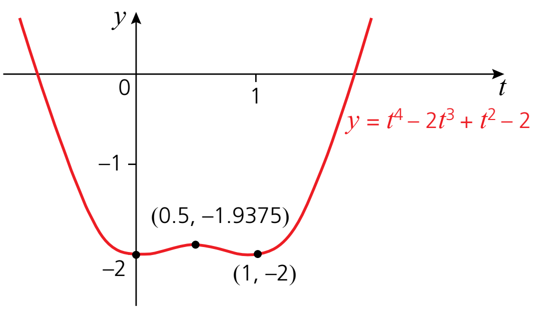
When $t = 0$, $y = (0)^4 - 2(0)^3 + (0)^2 - 2 = -2$.

When $t = 0.5$, $y = (0.5)^4 - 2(0.5)^3 + (0.5)^2 - 2 = -1.9375$.

When $t = 1$, $y = (1)^4 - 2(1)^3 + (1)^2 - 2 = -2$.

Plotting these points suggests that (0.5, –1.9375) is a maximum turning point and (0, –2) and (1, –2) are minima, but you need more information to be sure. For example when $t = -1$, $y = +2$ and when $t = 2$, $y = +2$ so you know that the curve goes above the horizontal axis on both sides.

You can find whether the gradient is positive or negative by taking a test point in each interval. For example, $t = 0.25$ in the interval $0 < t < 0.5$; when $t = 0.25$, $\frac{dy}{dt}$ is positive.



Exercise 14.2

You can use a graphic calculator to check your answers.

For each curve in questions 1 – 8:

i find $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$
ii classify the point(s) on the curve with these x-values
iii find the corresponding y-value(s)
iv sketch the curve.

1 $y = 1 + x - 2x^2$

2 $y = 12x + 3x^2 - 2x^3$

3 $y = x^3 - 4x^2 + 9$

4 $y = x^2 (x - 1)^2$

5 $y = x^4 - 8x^2 + 4$

6 $y = x^3 - 48x$

7 $y = x^3 + 6x^2 - 36x + 25$

8 $y = 2x^3 - 15x^2 + 24x + 8$

9 The graph of $y = px + qx^2$ passes through the point (3, –15). Its gradient at that point is –14.
   a Find the values of p and q.
   b Calculate the maximum value of y and state the value of x at which it occurs.

10 a Find the stationary points of the function $f(x) = x^2(3x^2 - 2x - 3)$ and distinguish between them.
   b Sketch the curve $y = f(x)$.

Using second derivatives

In the same way as $\frac{dy}{dx}$ or $f'(x)$ is the gradient of the curve $y = f(x)$, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $f''(x)$ represents the gradient of the curve $y = f'(x)$.

This is also written as $\frac{d^2y}{dx^2}$ and is called the **second derivative**. You can find it by differentiating the function $\frac{dy}{dx}$.

Note that $\frac{d^2y}{dx^2}$ is not the same as $\left(\frac{dy}{dx}\right)^2$.

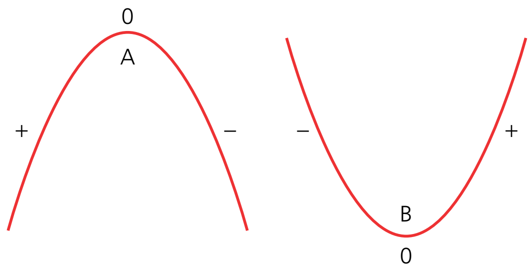
Worked example

Find $\frac{d^2y}{dx^2}$ for $y = 4x^3 + 3x - 2$.

Solution

$$\frac{dy}{dx} = 12x^2 + 3 \Rightarrow \frac{d^2y}{dx^2} = 24x$$

In many cases, you can use the second derivative to determine if a stationary point is a maximum or minimum instead of looking at the value of $\frac{dy}{dx}$ on either side of the turning point.



$\frac{d^2y}{d^2x} < 0$ at A

$\frac{d^2y}{dx^2} > 0$ at B

At A, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ showing that the gradient is zero and since $\frac{d^2y}{dx^2} < 0$, it is decreasing near that point, so must be going from positive to negative. This shows that A is a maximum turning point.

At B, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ showing that the gradient is zero and since $\frac{d^2y}{dx^2} > 0$, it is increasing near that point, so must be going from negative to positive. This shows that B is a minimum turning point.

Note that if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at the same point, you cannot make a decision about the type of turning point using this method.

Worked example

For $y = 2x^3 - 3x^2 - 12x + 4$

a Find $\frac{dy}{dx}$ and find the values of x when $\frac{dy}{dx} = 0$.

b Find the value of $\frac{d^2y}{dx^2}$ at each stationary point and hence determine its nature.

c Find the value of y at each of the stationary points.

d Sketch the curve $y = 2x^3 - 3x^2 - 12x + 4$.

Solution

a $\frac{dy}{dx} = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$

$= 6(x + 1)(x - 2)$

So $\frac{dy}{dx} = 0$ when $x = -1$ and when $x = 2$.

b $\frac{d^2y}{dx^2} = 12x - 6$

When $x = -1$, $\frac{d^2y}{dx^2} = -18 \Rightarrow$ a maximum

When $x = 2$, $\frac{d^2y}{dx^2} = 18 \Rightarrow$ a minimum

c When $x = -1$, $y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 4$

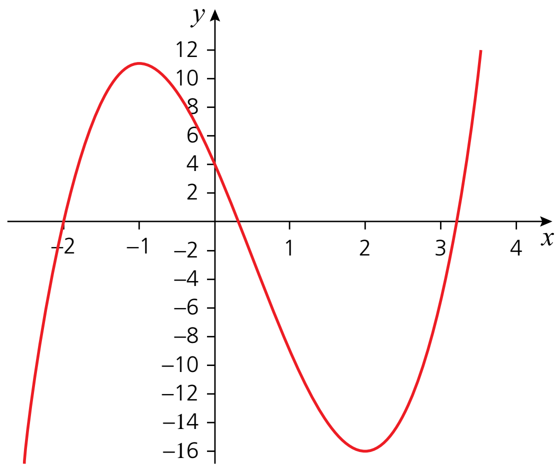
$= 11$

When $x = 2$, $y = 2(2)^3 - 3(2)^2 - 12(2) + 4$

$= -16$

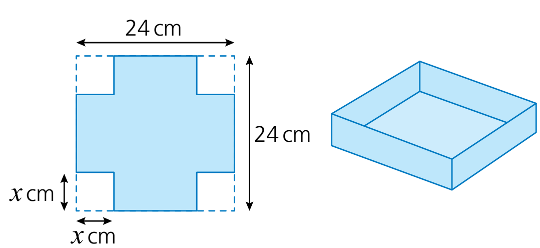
d The curve has a maximum turning point at $(-1, 11)$ and a minimum turning point at $(2, -16)$.

When $x = 0$, $y = 4$, so the curve crosses the y-axis at $(0, 4)$.



Worked example

Maria has made some sweets as a gift and makes a small box for them from a square sheet of card of side 24 cm. She cuts four identical squares of side x cm, one from each corner, and turns up the sides to make the box, as shown in the diagram.



a Write down an expression for the volume V of the box in terms of x.

b Find $\frac{dV}{dx}$ and the values of x when $\frac{dV}{dx} = 0$.

c Comment on this result.

d Find $\frac{d^2V}{dx^2}$ and hence find the depth when the volume is a maximum.

Solution

a The base of the box is a square of side $(24 - 2x)$ cm and the height is x cm, so

$V = (24 - 2x)^2 \times x$

Taking a factor of 2 out of each bracket → $= 4x(12 - x)^2 \text{ cm}^3$

b $V = 4x(144 - 24x + x^2)$
$= 576x - 96x^2 + 4x^3$

So $\frac{dV}{dx} = 576 - 192x + 12x^2$
$= 12(48 - 16x + x^2)$
$= 12(12 - x)(4 - x)$

So $\frac{dV}{dx} = 0$ when $x = 12$ and when $x = 4$.

c When $x = 12$ there is no box, since the piece of cardboard was only a square of side 24 cm.

Using $\frac{dV}{dx} = 576 - 192x + 12x^2$ →

d $\frac{d^2V}{dx^2} = -192 + 24x$

When $x = 4$, $\frac{d^2V}{dx^2} = -96$ which is negative.

Therefore the volume is a maximum when the depth $x = 4$ cm.

Exercise 14.3

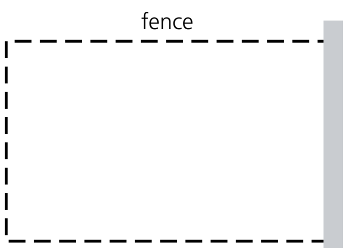
1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following functions:

a $y = x^3 - 3x^2 + 2x - 6$
b $y = 3x^4 - 4x^3$
c $y = x^5 - 5x + 1$

2 For each of the following curves
i find any stationary points
ii use the second derivative test to determine their nature.
a $y = 2x^2 - 3x + 4$
b $y = x^3 - 2x^2 + x + 6$
c $y = 4x^4 - 2x^2 + 1$
d $y = x^5 - 5x$

3 For $y = 2x^3 - 3x^2 - 36x + 4$
a Find $\frac{dy}{dx}$ and the values of x when $\frac{dy}{dx} = 0$.
b Find the value of $\frac{d^2y}{dx^2}$ at each stationary point and hence determine its nature.
c Find the value of y at each stationary point.
d Sketch the curve.

4 A farmer has 160 m of fencing and wants to use it to form a rectangular enclosure next to a barn.



Find the maximum area that can be enclosed and give its dimensions.

5 A cylinder has a height of h metres and a radius of r metres where $h + r = 3$.

a Find an expression for the volume of the cylinder in terms of r.
b Find the maximum volume.

6 A rectangle has sides of length x cm and y cm.
a If the perimeter is 24 cm, find the lengths of the sides when the area is a maximum, confirming that it is a maximum.
b If the area is 36 cm^2, find the lengths of the sides when the perimeter is a minimum, confirming that it is a minimum.

Equations of tangents and normals

Now that you know how to find the gradient of a curve at any point, you can use this to find the equation of the tangent at any given point on the curve.

Worked example

a Find the equation of the tangent to the curve $y = 3x^2 - 5x - 2$ at the point $(1, -4)$.

b Sketch the curve and show the tangent on your sketch.

Solution

a $y = 3x^2 - 5x - 2 \Rightarrow \frac{dy}{dx} = 6x - 5$

At $(1, -4)$, $\frac{dy}{dx} = 6 \times 1 - 5$

and so $m = 1$

Substituting $x = 1$ into this gradient function gives the gradient of the curve and therefore the tangent at this point.

So the equation of the tangent is given by

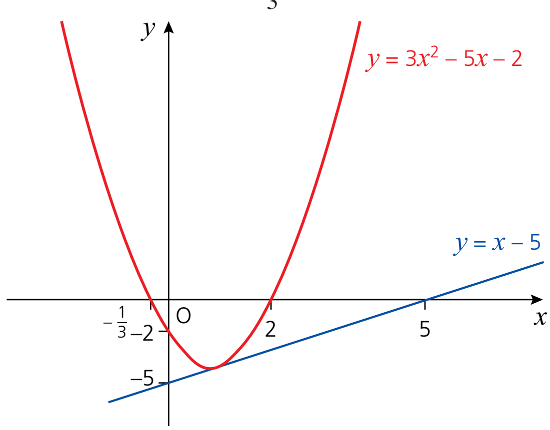
$$y - y_1 = (x - x_1)$$
$$y - (-4) = 1(x - 1)$$
$$\Rightarrow \quad y = x - 5$$

$x_1 = 1$, $y_1 = 4$ and $m = 1$

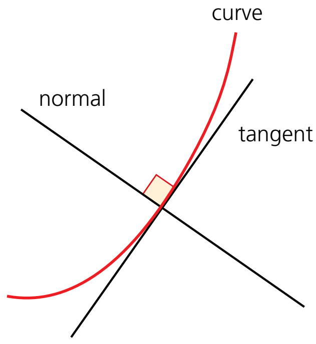
This is the equation of the tangent.

b $y = 3x^2 - 5x - 2$ is a ∪-shaped quadratic curve that crosses the crosses the y-axis when $y = -2$ and x-axis when $3x^2 - 5x - 2 = 0$.

$$3x^2 - 5x - 2 = 0 \Rightarrow (3x + 1)(x - 2) = 0$$
$$\Rightarrow x = -\frac{1}{3} \text{ or } x = 2$$



The **normal** to a curve at given point is the straight line that is at right angles to the tangent at that point, as shown below.



Remember that for perpendicular lines

$$m_1 m_2 = -1.$$

Worked example

Find the equation of the tangent and normal to the curve $y = 4x^2 - 2x^3$ at the point $(1, 2)$.

Draw a diagram showing the curve, the tangent and the normal.

It is slightly easier to use $y - y_1 = m(x - x_1)$ here than $y = mx + c$. If you substitute the gradient $m = 2$ and the point $(1, 2)$ into $y = mx + c$, you get $2 = 2 \times 1 + c$ and so $c = 0$ So the equation of the tangent is $y = 2x$.

Solution

$$y = 4x^2 - 2x^3 \quad \Rightarrow \quad \frac{dy}{dx} = 8x - 6x^2$$

At $(1, 2)$, the gradient is $\frac{dy}{dx} = 8 - 6 = 2$

The gradient of the tangent is $m_1 = 2$

So, using $\quad y - y_1 = m(x - x_1)$

the equation of the tangent is $y - 2 = 2(x - 1)$

$$y = 2x$$

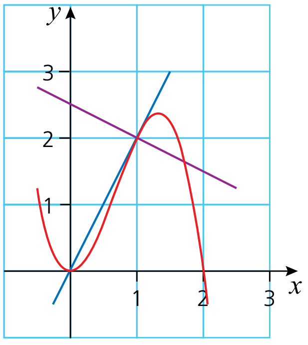
The gradient of the normal is $m_2 = -\frac{1}{m_1} = -\frac{1}{2}$

So, using $\quad y - y_1 = m(x - x_1)$

the equation of the normal is $y - 2 = -\frac{1}{2}(x - 1)$

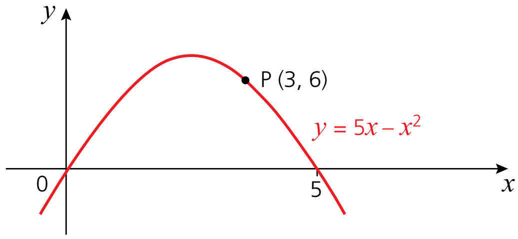
$$y = -\frac{x}{2} + \frac{5}{2}.$$

The curve, tangent and normal are shown on this graph.



Exercise 14.4

1 The sketch graph shows the curve of $y = 5x - x^2$.
The marked point, P, has coordinates (3, 6).



Find:

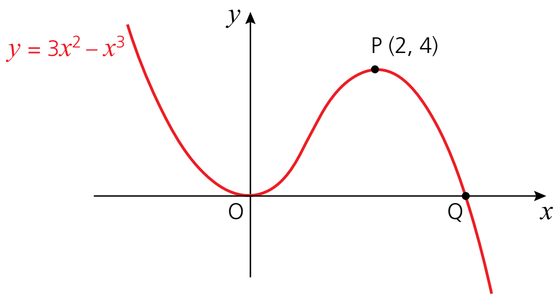
a the gradient function $\frac{dy}{dx}$

b the gradient of the curve at P

c the equation of the tangent at P

d the equation of the normal at P.

2 The sketch graph shows the curve of $y = 3x^2 - x^3$. The marked point, P, has coordinates (2, 4).



a Find:

i the gradient function $\frac{dy}{dx}$

ii the gradient of the curve at P

iii the equation of the tangent at P

iv the equation of the normal at P.

b The graph touches the x-axis at the origin O and crosses it at the point Q. Find:

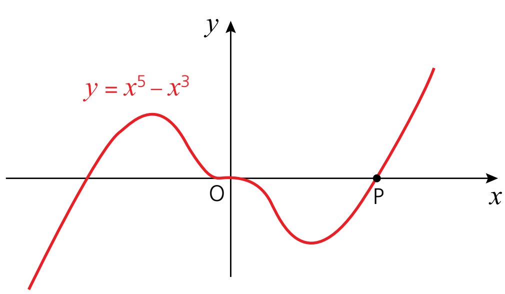
i the coordinates of Q

ii the gradient of the curve at Q

iii the equation of the tangent at Q.

c Without further calculation, state the equation of the tangent to the curve at O.

3 The sketch graph shows the curve of $y = x^5 - x^3$.



Exercise 14.4 (cont)

Find:

a the coordinates of the point P where the curve crosses the positive x-axis

b the equation of the tangent at P

c the equation of the normal at P.

The tangent at P meets the y-axis at Q and the normal meets the y-axis at R.

d Find the coordinates of Q and R and hence find the area of triangle PQR.

4 **a** Given that $f(x) = x^3 - 3x^2 + 4x + 1$, find $f'(x)$.

b The point P is on the curve $y = f(x)$ and its x-coordinate is 2.

i Calculate the y-coordinate of P.

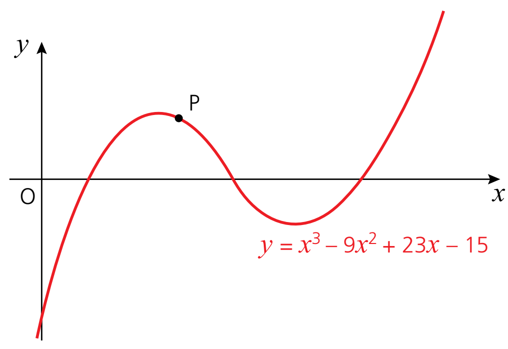
ii Find the equation of the tangent at P.

iii Find the equation of the normal at P.

c Find the values of x for which the curve has a gradient of 13.

5 The sketch graph shows the curve of $y = x^3 - 9x^2 + 23x - 15$.

The point P marked on the curve has its x-coordinate equal to 2.



Find:

a the gradient function $\frac{dy}{dx}$

b the gradient of the curve at P

c the equation of the tangent at P

d the coordinates of another point on the curve, Q, at which the tangent is parallel to the tangent at P

e the equation of the tangent at Q.

6 The point $(2, -8)$ is on the curve $y = x^3 - px + q$.

a Use this information to find a relationship between p and q.

b Find the gradient function $\frac{dy}{dx}$.

The tangent to this curve at the point $(2, -8)$ is parallel to the x-axis.

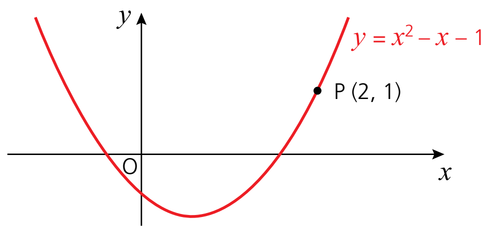
c Use this information to find the value of p.

d Find the coordinates of the other point where the tangent is parallel to the x-axis.

e State the coordinates of the point P where the curve crosses the y-axis.

f Find the equation of the normal to the curve at the point P.

7 The sketch graph shows the curve of $y = x^2 - x - 1$.



a Find the equation of the tangent at the point P(2, 1).
The normal at a point Q on the curve is parallel to the tangent at P.
b State the gradient of the tangent at Q.
c Find the coordinates of the point Q.

8 A curve has the equation $y = (x - 3)(7 - x)$.
a Find the gradient function $\frac{dy}{dx}$.
b Find the equation of the tangent at the point (6, 3).
c Find the equation of the normal at the point (6, 3).
d Which one of these lines passes through the origin?

9 A curve has the equation $y = 1.5x^3 - 3.5x^2 + 2x$.
a Show that the curve passes through the points (0, 0) and (1, 0).
b Find the equations of the tangents and normals at each of these points.
c Prove that the four lines in **b** form a rectangle.

Deriving these results from first principles is beyond the scope of this book.

Differentiating other functions of x

So far you have differentiated polynomials and other powers of x. Now this is extended to other expressions, starting with the three common trigonometrical functions. When doing this you will use the standard results that follow.

sin x, cos x and tan x

$$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$$

$$y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$$

$$y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$$

Recall $\sec x = \frac{1}{\cos x}$

When differentiating any trigonometric function, the angle must be in **radians**.

Worked example

Differentiate each of the following functions:

a $y = \sin x - \cos x$

b $y = 2\tan x + 3$

Solution

Using the results above

a $\frac{dy}{dx} = \cos x - (-\sin x)$

$= \cos x + \sin x$

b $y = 2\tan x + 3 \Rightarrow \frac{dy}{dx} = 2(\sec^2 x) + 0$

Differentiating a constant always gives zero.

$= 2\sec^2 x$

Worked example

a Sketch the graph of $y = \sin\theta$ for $0 \le \theta \le 2\pi$.

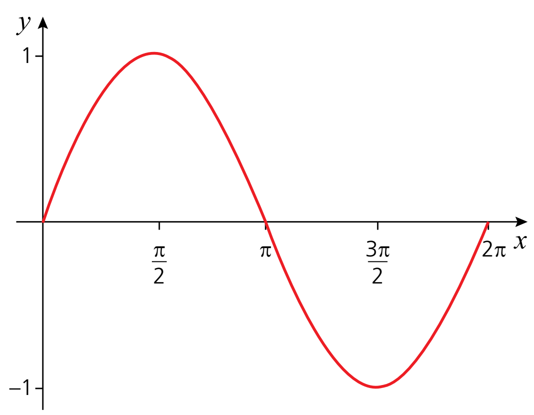
b **i** Find the value of $\frac{dy}{d\theta}$ when $\theta = \frac{\pi}{2}$.

ii At which other point does $\frac{dy}{d\theta}$ have this value?

c Use differentiation to find the value of $\frac{dy}{d\theta}$ when $\theta = \pi$.

Solution

a



b **i** The tangent to the curve when $\theta = \frac{\pi}{2}$ is horizontal, so $\frac{dy}{d\theta} = 0$.

ii The gradient is also 0 when $\theta = \frac{3\pi}{2}$.

c $y = \sin\theta \Rightarrow \frac{dy}{d\theta} = \cos\theta$

When $\theta = \pi$, $\frac{dy}{d\theta} = \cos\pi = -1$.

Worked example

a Find the turning point of the curve $y = \sin x - \cos x$ and determine its nature.

This means decide if it is a maximum or minimum point.

b Sketch the curve for $0 \leqslant x \leqslant \pi$.

Solution

a $y = \sin x - \cos x \Rightarrow \dfrac{dy}{dx} = \cos x + \sin x$

At the turning points $\cos x + \sin x = 0$

$\Rightarrow \quad \sin x = -\cos x$ ← Divide by $\cos x$

$\Rightarrow \quad \tan x = -1$

$\Rightarrow \quad x = -\dfrac{\pi}{4}$ (not in the required range)

or $\quad x = \dfrac{3\pi}{4}$

When $x = \dfrac{3\pi}{4}$, $y = \sin\dfrac{3\pi}{4} - \cos\dfrac{3\pi}{4}$

$= \sqrt{2}$

The turning point is at $\left(\dfrac{3\pi}{4}, \sqrt{2}\right)$.

When $x = \dfrac{\pi}{2}$ (to the left), $y = \sin\dfrac{\pi}{2} - \cos\dfrac{\pi}{2} = 1$.

When $x = \pi$ (to the right), $y = \sin\pi - \cos\pi = 1$. ← $1 < \sqrt{2}$

So the point $\left(\dfrac{3\pi}{4}, \sqrt{2}\right)$ is a maximum turning point.

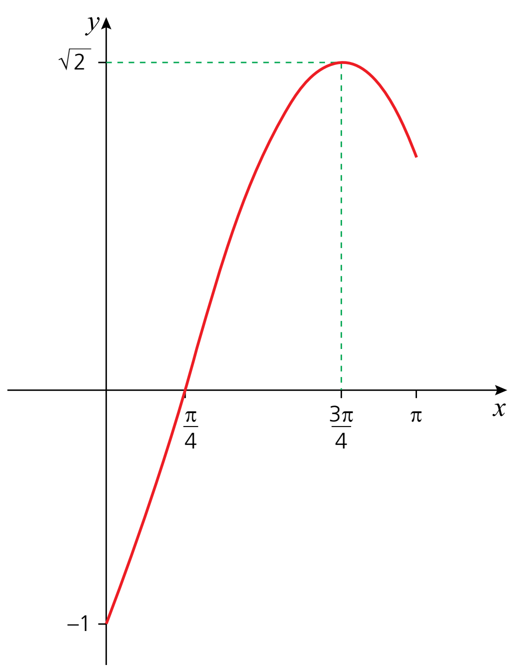
b When $x = 0$, $y = \sin 0 - \cos 0 = -1$. ← Check where the curve crosses the axes.

When $y = 0$, $0 = \sin x - \cos x$

$\Rightarrow \sin x = \cos x$ Divide by $\cos x$

$\Rightarrow \tan x = 1$

$\Rightarrow \quad x = \dfrac{\pi}{4}$



Worked example

For the curve $y = 2\cos\theta$ find:

a the equation of the tangent at the point where $\theta = \frac{\pi}{3}$

b the equation of the normal at the point where $\theta = \frac{\pi}{3}$.

Solution

a $y = 2\cos\theta \quad \Rightarrow \quad \frac{dy}{d\theta} = -2\sin\theta$

When $\theta = \frac{\pi}{3}$, $y = 2\cos\frac{\pi}{3}$
$= 1$

and $\frac{dy}{d\theta} = -2\sin\frac{\pi}{3}$
$= -\sqrt{3}$

Using $y = mx + c$ →

So the equation of the tangent is given by $y = -\theta\sqrt{3} + c$.

Substituting values for y and θ:

$$1 = -\left(\frac{\pi}{3}\right)\sqrt{3} + c \qquad \Rightarrow c = 1 + \frac{\pi\sqrt{3}}{3}$$

The equation of the tangent is therefore

$$y = -\theta\sqrt{3} + 1 + \frac{\pi\sqrt{3}}{3}.$$

b The gradient of the normal $= -1 \div \frac{dy}{d\theta}$

$$= -1 \div \left(-\sqrt{3}\right)$$
$$= \frac{1}{\sqrt{3}}$$

Using $y = mx + c$ →

The equation of the normal is given by $y = \frac{1}{\sqrt{3}}\theta + c$.

Substituting values for y and θ:

$$1 = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}\right) + c \qquad \Rightarrow \qquad c = 1 - \frac{\pi}{3\sqrt{3}}$$
$$= 1 - \frac{\pi\sqrt{3}}{9}$$

The equation of the normal is therefore

$$y = \frac{1}{\sqrt{3}}\theta + 1 - \frac{\pi}{3\sqrt{3}}.$$

Again, deriving these results from first principles is beyond the scope of this book.

e^x and ln x

You met exponential and logarithmic functions in Chapter 7. Here are the standard results for differentiating them.

$$y = e^x \Rightarrow \frac{dy}{dx} = e^x$$

$$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

This is the only function where $y = \frac{dy}{dx}$.

Worked example

Differentiate each of the following functions:

a $y = 5\ln x$

b $y = \ln(5x)$

c $y = 2e^x + \ln(2x)$

Solution

a $y = 5\ln x \Rightarrow \frac{dy}{dx} = 5\left(\frac{1}{x}\right)$

$= \frac{5}{x}$

b $y = \ln(5x) \Rightarrow y = \ln 5 + \ln x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x}$

ln 5 is a number so differentiating it gives zero.

c $y = 2e^x + \ln(2x) \Rightarrow \frac{dy}{dx} = 2e^x + \frac{1}{x}$

Part **b** shows an important result. Since $\ln(ax) = \ln a + \ln x$ for all values where $a > 0$,

$$y = \ln(ax) \Rightarrow \frac{dy}{dx} = \frac{1}{x}.$$

Worked example

a Find the turning point of the curve $y = 2x - \ln x$ and determine its nature.

b Sketch the curve for $0 < x \leqslant 3$.

Solution

a $y = 2x - \ln x \Rightarrow \frac{dy}{dx} = 2 - \frac{1}{x}$

$\frac{dy}{dx} = 0 \Rightarrow 2 = \frac{1}{x}$

$\Rightarrow x = 0.5$

When $x = 0.5$, $2x - \ln x = 1.7$ (1 d.p.).

When $x = 0.3$ (to the left), $2x - \ln x = 1.8$ (1 d.p.).

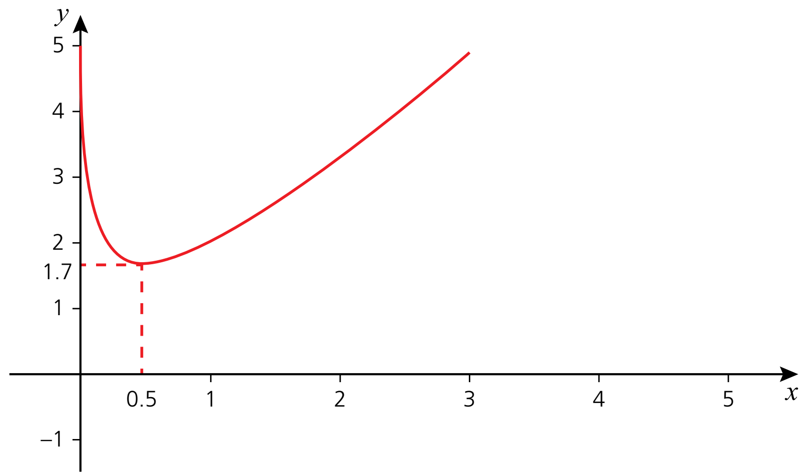
When $x = 1.0$ (to the right), $2x - \ln x = 2$ (1 d.p.).

Therefore the point (0.5, 1.7) is a minimum turning point.

Note

In this graph the y-axis is an asymptote. The curve gets nearer and nearer to it but never quite reaches it.

b



Notice that $\ln x$ is not defined for $x \leqslant 0$, and as $x \to 0$, $\ln x \to -\infty$ so $2x - \ln x \to +\infty$.

Worked example

For the curve $y = 2e^x + 5$ find the equation of:

a the tangent at the point where $x = -1$

b the normal at the point where $x = -1$.

Solution

a $y = 2e^x + 5 \quad \Rightarrow \quad \frac{dy}{dx} = 2e^x$

When $x = -1$, $y = 2e^{-1} + 5$

$= \frac{2}{e} + 5$

$\frac{dy}{dx} = 2e^{-1}$

Using $y = mx + c$ → So the equation of the tangent is given by $y = 2e^{-1}x + c$.

Substituting values for y and x:

$\frac{2}{e} + 5 = 2e^{-1}(-1) + c$

$= -\frac{2}{e} + c$

$\Rightarrow c = \frac{4}{e} + 5$

The equation of the tangent is therefore

$y = \frac{2}{e}x + \frac{4}{e} + 5.$

b The gradient of the normal $= -1 \div \frac{dy}{dx}$

$$= -1 \div \left(\frac{2}{e}\right)$$

$$= -\frac{e}{2}$$

Using $y = mx + c$ → The equation of the normal is given by $y = -\frac{e}{2}x + c$.

Substituting values for y and x:

$$\frac{2}{e} + 5 = -\frac{e}{2}(-1) + c$$

$$\Rightarrow c = \frac{2}{e} + 5 - \frac{e}{2}$$

The equation of the normal is therefore

$$y = -\frac{e}{2}x + \frac{2}{e} + 5 - \frac{e}{2}.$$

Differentiating products and quotients of functions

Deriving these results from first principles is beyond the scope of this book.

Sometimes you meet functions like $y = x^2e^x$, which are the product of two functions, x^2 and e^x. To differentiate such functions you use the **product rule**.

When $u(x)$ and $v(x)$ are two functions of x

» $y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

A shorthand form of $y = u(x) \times v(x)$

Worked example

Differentiate $y = (x^2 + 1)(2x - 3)$

a by expanding the brackets

b by using the product rule.

Solution

a $y = (x^2 + 1)(2x - 3)$

$$= 2x^3 - 3x^2 + 2x - 3$$

$$\Rightarrow \frac{dy}{dx} = 6x^2 - 6x + 2$$

b Let $u = (x^2 + 1)$ and $v = (2x - 3)$

$$\frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = 2$$

Product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

So $\frac{dy}{dx} = (x^2 + 1)(2) + (2x - 3)(2x)$

$$= 2x^2 + 2 + 4x^2 - 6x$$

$$= 6x^2 - 6x + 2$$

In this example you had a choice of methods; both gave you the same answer. In the next example there is no choice; you must use the product rule.

Worked example

Differentiate each of the following functions:

a $y = x^2 e^x$ **b** $y = x^3 \sin x$ **c** $y = (2x^3 - 4)(e^x - 1)$

Solution

a Let $u = x^2$ and $v = e^x$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = e^x$

Product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

So $\frac{dy}{dx} = x^2 e^x + e^x(2x)$

$= xe^x(x+2)$

b Let $u = x^3$ and $v = \sin x$

$\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = \cos x$

Product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

So $\frac{dy}{dx} = x^3 \cos x + (\sin x)(3x^2)$

$= x^2(x \cos x + 3 \sin x)$

c Let $u = (2x^3 - 4)$ and $v = (e^x - 1)$

$\frac{du}{dx} = 6x^2$ and $\frac{dv}{dx} = e^x$

Product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

So $\frac{dy}{dx} = (2x^3 - 4)(e^x) + (e^x - 1)(6x^2)$

$= 2x^3 e^x - 4e^x + 6x^2 e^x - 6x^2$

Sometimes you meet functions like $y = \frac{e^x}{x^2 + 1}$ where one function, in this case e^x, is divided by another, $x^2 + 1$. To differentiate such functions you use the **quotient rule**.

For $y = \frac{u}{v}$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Worked example

Differentiate $y = \frac{x^3 + 3}{2x^2}$

a by simplifying first **b** by using the quotient rule.

Solution

a $y = \dfrac{x^3+3}{x^2}$

$= (x^3+3)x^{-2}$

$= x + 3x^{-2}$

So $\dfrac{dy}{dx} = 1 - 6x^{-3}$

b Let $u = x^3 + 3$ and $v = x^2$

$\dfrac{du}{dx} = 3x^2$ and $\dfrac{dv}{dx} = 2x$

Quotient rule: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{x^2(3x^2)-(x^3+3)2x}{(x^2)^2}$

$= \dfrac{3x^4 - 2x^4 - 6x}{x^4}$

$= \dfrac{x^4 - 6x}{x^4}$

$= 1 - 6x^{-3}$

This quotient rule is longer in this case, but is useful when it is not possible to simplify first.

Worked example

Differentiate each of the following functions:

a $y = \dfrac{2x^3+3}{x^2-1}$

b $y = \dfrac{e^x}{x^2}$

Solution

a $y = \dfrac{2x^3+3}{x^2-1}$

Let $u = 2x^3 + 3$ and $v = x^2 - 1$

$\dfrac{du}{dx} = 6x^2$ and $\dfrac{dv}{dx} = 2x$

Quotient rule: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{(x^2-1)6x^2 - (2x^3+3)2x}{(x^2-1)^2}$

$= \dfrac{6x^4 - 6x^2 - 4x^4 - 6x}{(x^2-1)^2}$

$= \dfrac{2x^4 - 6x^2 - 6x}{(x^2-1)^2}$

$= \dfrac{2x(x^3 - 3x - 3)}{(x^2-1)^2}$

b $y = \dfrac{e^x}{x^2}$

Let $u = e^x$ and $v = x^2$

$\dfrac{du}{dx} = e^x$ and $\dfrac{dv}{dx} = 2x$

Quotient rule: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{x^2(e^x) - e^x(2x)}{(x^2)^2}$

$= \dfrac{x^2e^x - 2xe^x}{(x^2)^2}$

$= \dfrac{xe^x(x-2)}{x^4}$

$= \dfrac{e^x(x-2)}{x^3}$

Differentiating composite functions

Sometimes you will need to differentiate an expression that is a function of a function.

For example, look at $y = \sqrt{x^2+1}$; the function 'Take the positive square root of', denoted by $\sqrt{\ }$, is applied to the function (x^2+1). In such cases you use the **chain rule**.

You know how to differentiate $y = \sqrt{x}$ and you know how to differentiate $y = x^2 + 1$ but so far you have not met the case where two functions like this are combined into one.

The first step is to make a substitution.

Let $u = x^2 + 1$.

So now you have to differentiate $y = \sqrt{u}$ or $y = u^{\frac{1}{2}}$.

You know the right-hand side becomes $\frac{1}{2}u^{\frac{1}{2}-1}$ or $\frac{1}{2}u^{-\frac{1}{2}}$.

What about the left-hand side?

The differentiation is with respect to u rather than x and so you get $\frac{dy}{du}$ rather than $\frac{dy}{dx}$ that you actually want.

To go from $\frac{dy}{du}$ to $\frac{dy}{dx}$, you use the **chain rule**,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

You made the substitution $u = x^2 + 1$ and differentiating this gives

$$\frac{du}{dx} = 2x.$$

So now you can substitute for both $\frac{dy}{du}$ and $\frac{du}{dx}$ in the chain rule and get

$$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times 2x = xu^{-\frac{1}{2}}$$

This isn't quite the final answer because the right-hand side includes the letter u whereas it should be given entirely in terms of x.

Substituting $u = x^2 + 1$, gives $\frac{dy}{dx} = x(x^2+1)^{-\frac{1}{2}}$ and this is now the answer.

However it can be written more neatly as

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}.$$

Notice how the awkward function, $\sqrt{x^2+1}$, has reappeared in the final answer.

This is an important method and with experience you will find short cuts that will mean you don't have to write everything out in full as it has been here.

Worked example

Given that $y = (2x - 3)^4$, find $\frac{dy}{dx}$.

Solution

Let $u = (2x - 3)$ so $y = u^4$

$$\frac{dy}{du} = 4u^3$$
$$= 4(2x - 3)^3$$
$$\frac{du}{dx} = 2$$

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ →

$$\Rightarrow \frac{dy}{dx} = 4(2x - 3)^3 \times 2$$
$$= 8(2x - 3)^3$$

You can use the chain rule in conjunction with the product rule or the quotient rule as shown in the following example.

Worked example

Find $\frac{dy}{dx}$ when $y = (2x + 1)(x + 2)^{10}$.

Solution

Let $u = (2x + 1)$ and $v = (x + 2)^{10}$

Using the chain rule to find $\frac{dv}{dx}$ →

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 10(x + 2)^9 \times 1$

Using the product rule $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ →

$$\frac{dy}{dx} = (2x + 1) \times 10(x + 2)^9 + (x + 2)^{10} \times 2$$
$$= 10(2x + 1)(x + 2)^9 + 2(x + 2)^{10}$$
$$= 2(x + 2)^9 [5(2x + 1) + (x + 2)]$$
$$= 2(x + 2)^9 (11x + 7)$$

← Taking $2(x + 2)^9$ out as a common factor

Exercise 14.5

1 Differentiate each of the following functions:

a $y = 3\sin x - 2\tan x$ **b** $y = 5\sin\theta - 6$ **c** $y = 2\cos\theta - 2\sin\theta$
d $y = 4\ln x$ **e** $y = \ln 4x$ **f** $y = 3e^x$
g $y = 2e^x - \ln 2x$

2 Use the product rule to differentiate each of the following functions:

a $y = x\sin x$ **b** $y = x\cos x$ **c** $y = x\tan x$
d $y = e^x\sin x$ **e** $y = e^x\cos x$ **f** $y = e^x\tan x$

3 Use the quotient rule to differentiate each of the following functions:

a $y = \frac{\sin x}{x}$ **b** $y = \frac{x}{\sin x}$ **c** $y = \frac{\cos x}{x^2}$
d $y = \frac{x^2}{\cos x}$ **e** $y = \frac{x}{\tan x}$ **f** $y = \frac{\tan x}{x}$

Exercise 14.5 (cont)

4 Use the chain rule to differentiate each of the following functions:

a $y=(x+3)^4$ b $y=(2x+3)^4$ c $y=\left(x^2+3\right)^4$

d $y=\sqrt{x+3}$ e $y=\sqrt{2x+3}$ f $y=\sqrt{x^2+3}$

5 Use an appropriate method to differentiate each of the following functions:

a $y=\dfrac{\sin x}{1+\cos x}$ b $y=\dfrac{1+\cos x}{\sin x}$

c $y=\sin x(1+\cos x)$ d $y=\cos x(1+\sin x)$

e $y=\sin x(1+\cos x)^2$ f $y=\cos x(1+\sin x)^2$

6 Use an appropriate method to differentiate each of the following functions:

a $y=\mathrm{e}^x\ln x$ b $y=\dfrac{\mathrm{e}^x}{\ln x}$ c $y=\dfrac{\ln x}{\mathrm{e}^x}$

7 Use an appropriate method to differentiate each of the following functions:

a $\mathrm{e}^{-x}\sin x$ b $y=\dfrac{\mathrm{e}^{-x}}{\sin x}$ c $y=\dfrac{\sin x}{\mathrm{e}^{-x}}$

8 A curve has the equation $y=\sin x-\cos x$ where x is measured in radians.

a Show that the curve passes through the points $(0,-1)$ and $(\pi,1)$.

b Find the equations of the tangents and normals at each of these points.

9 A curve has the equation $y=2\tan x-1$ where x is measured in radians.

a Show that the curve passes through the points $(0,-1)$ and $\left(\frac{\pi}{4},1\right)$.

b Find the equations of the tangents and normals at each of these points.

10 A curve has the equation $y=2\ln x-1$.

a Show that the curve passes through the point $(e,1)$

b Find the equations of the tangent and normal at this point.

11 A curve has the equation $y=\mathrm{e}^x-\ln x$.

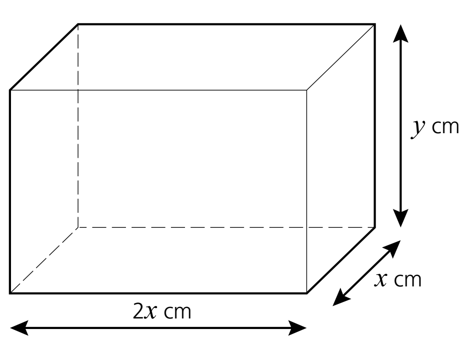
a Sketch the curves $y=\mathrm{e}^x$ and $y=\ln x$ on the same axes and explain why this implies that $\mathrm{e}^x-\ln x$ is always positive.

b Show that the curve $y=\mathrm{e}^x-\ln x$ passes through the point $(1,e)$.

c Find the equations of the tangent and normal at this point.

Past-paper questions

1 The diagram shows a cuboid with a rectangular base of sides x cm and $2x$ cm. The height of the cuboid is y cm and its volume is 72 cm³.



(i) Show that the surface area A cm^2 of the cuboid is given by $A = 4x^2 + \frac{216}{x}$. [3]

(ii) Given that x can vary, find the dimensions of the cuboid when A is a minimum. [4]

(iii) Given that x increases from 2 to $2 + p$, where p is small, find, in terms of p, the corresponding approximate change in A, stating whether this change is an increase or a decrease. [3]

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2 Find $\frac{dy}{dx}$ when

(i) $y = \cos 2x \sin\left(\frac{x}{3}\right)$, [4]

(ii) $y = \frac{\tan x}{1 + \ln x}$. [4]

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3 **(a)** Differentiate $4x^3 \ln(2x + 1)$ with respect to x. [3]

(b) **(i)** Given that $y = \frac{2x}{\sqrt{x+2}}$, show that $\frac{dy}{dx} = \frac{x+4}{(\sqrt{x+2})^3}$. [4]

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(Part question: part b (ii) and (iii) omitted)
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Paper 12 November 2013
(Part question: part b (ii) and (iii) omitted)

Learning outcomes

Now you should be able to:

★ understand the idea of a derived function

★ use the notations $f'(x)$, $f''(x)$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ $\left[= \frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$

★ apply differentiation to gradients, tangents and normals, stationary points, connected rates of change, small increments and approximations and practical maxima and minima problems

★ use the first and second derivative tests to discriminate between maxima and minima

★ use the derivatives of the standard functions x^n (for any rational n), $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$, together with constant multiples, sums and composite functions of these

★ differentiate products and quotients of functions.

Key points

- ✔ $y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = nkx^{n-1}$ and $y = c \quad \Rightarrow \quad \frac{dy}{dx} = 0$,
  where n is a positive integer and k and c are constants.
- ✔ $y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x)$
- ✔ $\frac{dy}{dx} = 0$ at a stationary point. The nature of the stationary point can be determined by looking at the sign of the gradient immediately either side of it or by considering the sign of $\frac{d^2y}{dx^2}$.
  - If $\frac{d^2y}{dx^2} < 0$, the point is a maximum.
  - If $\frac{d^2y}{dx^2} > 0$, the point is a minimum.
  - If $\frac{d^2y}{dx^2} = 0$, the point could be a maximum, a minimum or a point of inflection. Check the values of $\frac{dy}{dx}$ on either side of the point to determine its nature.
- ✔ For the tangent and normal at (x_1, y_1)
  - the gradient of the tangent, $m_1 = \frac{dy}{dx}$
  - the gradient of the normal, $m_2 = -\frac{1}{m_1}$
  - the equation of the tangent is $y - y_1 = m_1(x - x_1)$
  - the equation of the normal is $y - y_1 = m_2(x - x_1)$.
- ✔ Derivatives of other functions:

Function	**Derivative $\frac{dy}{dx}$**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

- ✔ The product rule $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.
- ✔ The quotient rule $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.
- ✔ The chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.