12 Series

I would like one grain of wheat to be put on the first square of my board, two on the second square, four on the third square, eight on the fourth and so on.

Attributed to Sissa ben Dahir (6th century)

Discussion point

The origin of the game of chess is uncertain, both in time and place. According to one legend it was invented by Sissa ben Dahir, Vizier to Indian king Shirham. The king asked Sissa ben Dahir what he would like for a reward, and his reply is quoted above. The king agreed without doing any calculations.

Given that one grain of wheat weighs about 50 mg, what mass of wheat would have been placed on the last square?

Definitions and notation

A **sequence** is a set of numbers in a given order, for example

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots.$$

Each of these numbers is called a **term** of the sequence. When the terms of a sequence are written algebraically, the position of any term in the sequence is usually shown by a subscript, so that a general sequence is written:

$u_1, u_2, u_3, \ldots$, with general term u_k.

For the previous sequence, the first term is $u_1 = \frac{1}{2}$, the second term is $u_2 = \frac{1}{4}$, and so on.

When the terms of a sequence are added together, for example,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

the resulting sum is called a **series**. The process of adding the terms together is called **summation** and indicated by the symbol $\sum$ (the Greek letter sigma), with the position of the first and last terms involved given as **limits**.

So $u_1 + u_2 + u_3 + u_4 + u_5$ is written $\sum_{k=1}^{k=5} u_k$ or $\sum_{k=1}^{5} u_k$.

In cases like this one, where there is no possibility of confusion, the sum is normally written more simply as $\sum_1^5 u_k$.

If all the terms are to be summed, it is usually denoted even more simply as $\sum_k u_k$, or even $\sum u_k$.

A sequence may have an infinite number of terms, in which case it is called an **infinite sequence**. The corresponding series is called an **infinite series**.

The phrase 'sum of a sequence' is often used to mean the sum of the terms of a sequence (i.e. the series).

Although the word **series** can describe the sum of the terms of any sequence in mathematics, it is usually used only when summing the sequence provides a useful or interesting overall result.

For example:

$$(1+t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4$$

This series has a finite number of terms (5).

$$\sqrt{11} = \frac{10}{3}\left[1 - \frac{1}{2}(0.01) - \frac{1}{8}(0.01)^2 - \frac{1}{16}(0.01)^3 \ldots\right]$$

This series has an infinite number of terms.

The binomial theorem

A special type of series is produced when a binomial (i.e. two-part) expression such as $(x + 1)$ is raised to a power. The resulting expression is often called a **binomial expansion**.

The simplest binomial expansion is $(x + 1)$ itself. This and other powers of $(x + 1)$ are given below.

Expressions like these, consisting of integer powers of x and constants are called polynomials.

$$(x+1)^1 = 1x + 1$$
$$(x+1)^2 = 1x^2 + 2x + 1$$
$$(x+1)^3 = 1x^3 + 3x^2 + 3x + 1$$
$$(x+1)^4 = 1x^4 + 4x^3 + 6x^2 + 4x + 1$$
$$(x+1)^5 = 1x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

If you look at the coefficients on the right-hand side you will see that they form a pattern.

These numbers are called binomial coefficients.

(1)
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

This is called **Pascal's triangle**, or the **Chinese triangle**. Each number is obtained by adding the two above it, for example

4 + 6
gives 10

This pattern of coefficients is very useful when you need to write down the expansions of other binomial expressions. For example,

Notice how in each term the sum of the powers of x and y is the same as the power of $(x + y)$.

$$(x+y) = 1x + 1y$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

This is a binomial expression.

These numbers are binomial coefficients.

Worked example

Write out the binomial expansion of $(a + 3)^5$.

Solution

The binomial coefficients for power 5 are 1 5 10 10 5 1.

In each term, the sum of the powers of a and 3 must equal 5.

So the expansion is:

$$1 \times a^5 + 5 \times a^4 \times 3 + 10 \times a^3 \times 3^2 + 10 \times a^2 \times 3^3 + 5 \times a \times 3^4 + 1 \times 3^5$$

i.e. $a^5 + 15a^4 + 90a^3 + 270a^2 + 405a + 243$.

Worked example

Write out the binomial expansion of $(3x - 2y)^4$.

Solution

The binomial coefficients for power 4 are 1 4 6 4 1.

The expression $(3x - 2y)$ is treated as $(3x + (-2y))$.

So the expansion is

$$1 \times (3x)^4 + 4 \times (3x)^3 \times (-2y) + 6 \times (3x)^2 \times (-2y)^2 + 4 \times (3x) \times (-2y)^3 + 1 \times (-2y)^4$$

i.e. $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$

Pascal's triangle (and the binomial theorem) had actually been discovered by Chinese mathematicians several centuries earlier, and can be found in the works of Yang Hui (around AD1270) and Chu Shi-kie (in AD1303). However, Pascal is remembered for his application of the triangle to elementary probability, and for his study of the relationships between binomial coefficients.

Tables of binomial coefficients

Values of binomial coefficients can be found in books of tables. It can be helpful to use these when the power becomes large, since writing out Pascal's triangle becomes progressively longer and more tedious, row by row. Note that since the numbers are symmetrical about the middle number, tables do not always give the complete row of numbers.

Worked example

Write out the full expansion of $(a + b)^8$.

Solution

The binomial coefficients for the power 8 are

1 8 28 56 70 56 28 8 1

and so the expansion is

$$a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8.$$

The formula for a binomial coefficient

You may need to find binomial coefficients that are outside the range of your tables. The tables may, for example, list the binomial coefficients for powers up to 20. What happens if you need to find the coefficient of x^{17} in the expansion of $(x + 2)^{25}$? Clearly you need a formula that gives binomial coefficients.

The first thing you need is a notation for identifying binomial coefficients. It is usual to denote the power of the binomial expression by n, and the position in the row of binomial coefficients by r, where r can take any value from 0 to n. So, for row 5 of Pascal's triangle

$n = 5$:	1	5	10	10	5	1
	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$

The general binomial coefficient corresponding to values of n and r is written as $\binom{n}{r}$. An alternative notation is ${}^nC_r$, which is said as 'N C R'.

Thus $\binom{5}{3} = {}^5C_3 = 10$.

Note that 0! is defined to be 1. You will see the need for this when you use the formula for $\binom{n}{r}$.

The next step is to find a formula for the general binomial coefficient $\binom{n}{r}$.

Real-world activity

The table shows an alternative way of laying out Pascal's triangle.

		Column (r)								
		0	**1**	**2**	**3**	**4**	**5**	**6**	**…**	**r**
Row (n)	**1**	1	1							
	2	1	2	1						
	3	1	3	3	1					
	4	1	4	6	4	1				
	5	1	5	10	10	5	1			
	6	1	6	15	20	15	6	1		
	…	…	…	…	…	…	…	…	…	
	…	…	…	…	…	…	…	…	…	…
	n	1	n	?	?	?	?	?	?	?

Show that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, by following the procedure below.

The numbers in column 0 are all 1.

To find each number in column 1 you multiply the 1 in column 0 by the row number, n.

1 Find, in terms of n, what you must multiply each number in column 1 by to find the corresponding number in column 2.

2 Repeat the process to find the relationship between each number in column 2 and the corresponding number in column 3.

3 Show that repeating the process leads to

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots \times r} \text{ for } r \geqslant 1.$$

4 Show that this can also be written as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

and that it is also true for $r = 0$.

Worked example

Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to calculate these.

a $\binom{7}{0}$ **b** $\binom{7}{1}$ **c** $\binom{7}{2}$ **d** $\binom{7}{3}$

e $\binom{7}{4}$ **f** $\binom{7}{5}$ **g** $\binom{7}{6}$ **h** $\binom{7}{7}$

Solution

a $\binom{7}{0} = \frac{7!}{0!(7-0)!} = \frac{5040}{1 \times 5040} = 1$

b $\binom{7}{1} = \frac{7!}{1!6!} = \frac{5040}{1 \times 720} = 7$

c $\binom{7}{2} = \frac{7!}{2!5!} = \frac{5040}{2 \times 120} = 21$

d $\binom{7}{3} = \frac{7!}{3!4!} = \frac{5040}{6 \times 24} = 35$

e $\binom{7}{4} = \frac{7!}{4!3!} = \frac{5040}{24 \times 6} = 35$

f $\binom{7}{5} = \frac{7!}{5!2!} = \frac{5040}{120 \times 21} = 21$

g $\binom{7}{6} = \frac{7!}{6!1!} = \frac{5040}{720 \times 1} = 7$

h $\binom{7}{7} = \frac{7!}{7!0!} = \frac{5040}{5040 \times 1} = 1$

Note

Most scientific calculators have factorial buttons, e.g. [x!]. Many also have [^nC_r] buttons. Find out how best to use your calculator to find binomial coefficients, as well as practising non-calculator methods.

Worked example

Find the coefficient of x^{19} in the expansion of $(x + 3)^{25}$.

Solution

$$(x+3)^{25} = \binom{25}{0}x^{25} + \binom{25}{1}x^{24}3^1 + \binom{25}{2}x^{23}3^2 + \ldots + \binom{25}{8}6x^{19}3^6 + \ldots \binom{25}{25}3^{25}$$

So the required term is $\binom{25}{6} \times x^{19}\,3^6$

$$\binom{25}{6} = \frac{25!}{6!19!} = \frac{25 \times 24 \times 23 \times 22 \times 21 \times 20 \times \cancel{19!}}{6! \times \cancel{19!}}$$

$$= 177\,100.$$

So the coefficient of x^{19} is $177\,100 \times 3^6 = 129\,105\,900$.

Notice how 19! was cancelled in working out $\binom{25}{6}$. Factorials become large numbers very quickly and you should keep a look-out for such opportunities to simplify calculations.

The expansion of $(1 + x)^n$

When deriving the result for $\binom{n}{r}$ you found the binomial coefficients in the form

$$1 \quad n \quad \frac{n(n-1)}{2!} \quad \frac{n(n-1)(n-2)}{3!} \quad \frac{n(n-1)(n-2)(n-3)}{4!} \quad \ldots$$

This form is commonly used in the expansion of expressions of the type $(1 + x)^n$.

The first few terms →

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{1\times 2} + \frac{n(n-1)(n-2)x^3}{1\times 2\times 3} + \frac{n(n-1)(n-2)(n-3)x^4}{1\times 2\times 3\times 4} + \ldots$$

The last few terms →

$$+ \frac{n(n-1)}{1\times 2}x^{n-2} + nx^{n-1} + 1x^n$$

Worked example

Use the binomial expansion to write down the first four terms, in ascending powers of x, of $(1 + x)^8$.

Solution

$$(1+x)^8 = 1 + 8x + \frac{8\times 7}{1\times 2}x^2 + \frac{8\times 7\times 6}{1\times 2\times 3}x^3 + \ldots$$

The power of x is the same as the largest number underneath.

Two numbers on top, two underneath.

Three numbers on top, three underneath.

$$= 1 + 8x + 28x^2 + 56x^3 + \ldots$$

An expression like $1 + 8x + 28x^2 + 56x^3$ ... is said to be in **ascending** powers of x, because the powers of x are increasing from one term to the next.

An expression like $x^8 + 8x^7 + 28x^6 + 56x^5$ ... is in **descending** powers of x, because the powers of x are decreasing from one term to the next.

Worked example

Use the binomial expansion to write down the first four terms, in ascending powers of x, of $(1 - 2x)^6$. Simplify the terms.

Solution

Think of $(1 - 2x)^6$ as $(1 + (-2x))^6$. Keep the brackets while you write out the terms.

$$(1+(-2x))^6 = 1 + 6(-2x) + \frac{6\times 5}{1\times 2}(-2x)^2 + \frac{6\times 5\times 4}{1\times 2\times 3}(-2x)^3 + \ldots$$

$$= 1 - 12x + 60x^2 - 160x^3 + \ldots$$

Notice how the signs alternate.

Exercise 12.1

1 Write out the following binomial expressions:
a $(1+x)^4$ **b** $(1+2x)^4$ **c** $(1+3x)^4$

2 Write out the following binomial expressions:
a $(2+x)^4$ **b** $(3+x)^4$ **c** $(4+x)^4$

3 Write out the following binomial expressions:
a $(x+y)^4$ **b** $(x+2y)^4$ **c** $(x+3y)^4$

4 Use a non-calculator method to calculate the following binomial coefficients. Check your answers using your calculator's shortest method.
a $\binom{5}{3}$ **b** $\binom{7}{2}$ **c** $\binom{7}{4}$
d $\binom{7}{5}$ **e** $\binom{5}{0}$ **f** $\binom{13}{3}$

5 Find the coefficients of the term shown for each expansion:
a x^4 in $(1+x)^6$ **b** x^5 in $(1+x)^7$ **c** x^6 in $(1+x)^8$

6 Find the first three terms, in ascending powers of x, in the expansion of $(3+kx)^5$.

7 Find the first three terms, in descending powers of x, in the expansion of $\left(3x-\frac{3}{x}\right)^6$.

8 **a** Simplify $(1+t)^3-(1-t)^3$.
b Show that $x^3-y^3=(x-y)(x^2+xy+y^2)$.
c Substitute $x=1+t$ and $y=1-t$ in the result in part **b** and show that your answer is the same as that for part **a**.

9 Find the coefficients of x^3 and x^4 for each of the following:
a $(1+x)(1-x)^6$ **b** $(1-x)(1+x)^6$

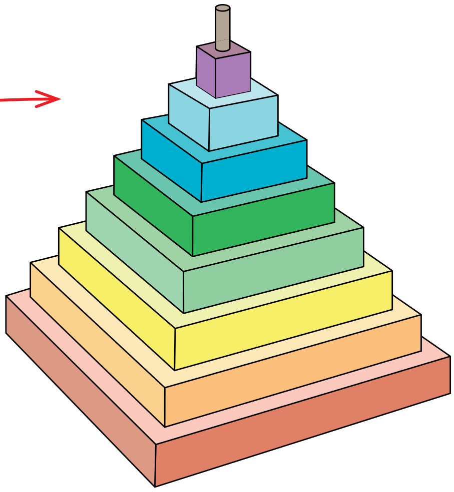
10 Write down the first four terms, in ascending powers of x, of the following binomial expressions:
a $(1-2x)^6$ **b** $(2-3x)^6$ **c** $(3-4x)^6$

11 Find the first four terms, in descending powers of x, of the following binomial expressions:
a $\left(x^2+\frac{1}{x}\right)^5$ **b** $\left(x^2-\frac{1}{x}\right)^5$ **c** $\left(x^3+\frac{1}{x}\right)^5$ **d** $\left(x^3-\frac{1}{x}\right)^5$

12 The first three terms in the expansion of $(2-ax)^n$ in ascending powers of x are 32, –240 and 720. Find the values of a and n.

Arithmetic progressions

The smallest square shape in this toy has sides 1 cm long, and the lengths of the sides increase in steps of 1 cm.



Any ordered set of numbers, like the areas of the squares in this toy, form a sequence. In mathematics, we are particularly interested in sequences with a well-defined pattern, often in the form of an algebraic formula linking the terms. The area of the squares in the toy, in cm^2, are $1^2, 2^2, 3^2, 4^2, \ldots$ or $1, 4, 9, 16 \ldots$.

A sequence in which the terms increase by the addition of a fixed amount (or decrease by the subtraction of a fixed amount) is described as an **arithmetic sequence** or **arithmetic progression (A. P.)**. The increase from one term to the next is called the **common difference**.

Thus the sequence $8 \quad 11 \quad 14 \quad 17\ldots$ is arithmetic with

$+3 \quad +3 \quad +3$

common difference 3. This sequence can be written algebraically as

$$u_k = 5 + 3k \text{ for } k = 1, 2, 3, \ldots$$

When $k = 1, u_1 = 5 + 3 = 8$

$k = 2, u_2 = 5 + 6 = 11$

$k = 3, u_3 = 5 + 9 = 14$ and so on.

(You can also write this as $u_k = 8 + 3(k - 1)$ for $k = 1, 2, 3, \ldots$.)

This version has the advantage that the right-hand side begins with the first term of the sequence.

As successive terms of an arithmetic progression increase (or decrease) by a fixed amount called the common difference, d, you can define each term in the sequence in relation to the previous term:

$$u_{k+1} = u_k + d.$$

When the terms of an arithmetic progression are added together, the sum is called an **arithmetic series**.

Notation

The following conventions are used in this book to describe arithmetic progressions and sequences:

- first term, $u_1 = a$
- number of terms = n
- last term, $u_n = l$
- common difference = d
- the general term, u_k, is that in position k (i.e. the kth term).

Thus in the arithmetic progression 7, 9, 11, 13, 15, 17, 19

$$a = 7, l = 19, d = 2 \text{ and } n = 7.$$

The terms are formed as follows:

$$\begin{aligned}
u_1 &= a & &= 7 \\
u_2 &= a + d & &= 7 + 2 = 9 \\
u_3 &= a + 2d & &= 7 + 2 \times 2 = 11 \\
u_4 &= a + 3d & &= 7 + 3 \times 2 = 13 \\
u_5 &= a + 4d & &= 7 + 4 \times 2 = 15 \\
u_6 &= a + 5d & &= 7 + 5 \times 2 = 17 \\
u_7 &= a + 6d & &= 7 + 6 \times 2 = 19
\end{aligned}$$

The 7th term is the 1st term (7) plus six times the common difference (2).

This shows that any term is given by the first term plus a number of differences. The number of differences is, in each case, one less than the number of the term. You can express this mathematically as

$$u_k = a + (k - 1)d.$$

For the last term, this becomes

$$l = a + (n - 1)d.$$

These are both general formulae so apply to any arithmetic progression.

Worked example

Find the 19th term in the arithmetic progression 20, 16, 12, …

Solution

In this case $a = 20$ and $d = -4$.

Using $u_k = a + (k - 1)d$, you obtain

$$\begin{aligned}
u_{19} &= 20 + (19 - 1) \times (-4) \\
&= 20 - 72 \\
&= -52.
\end{aligned}$$

The 19th term is –52.

Worked example

How many terms are there in the sequence 12, 16, 20, …, 556?

Solution

This is an arithmetic sequence with first term $a = 12$, last term $l = 556$ and common difference $d = 4$.

Using the result $l = a + (n-1)d$, you have

$$\begin{aligned} 556 &= 12 + 4(n-1) \\ \Rightarrow \quad 4n &= 556 - 12 + 4 \\ \Rightarrow \quad n &= 137 \end{aligned}$$

There are 137 terms.

Note

The relationship $l = a + (n-1)d$ may be rearranged to give

$$n = \frac{l-a}{d} + 1$$

This gives the number of terms in an A.P. directly if you know the first term, the last term and the common difference.

The sum of the terms of an arithmetic progression

When Carl Friederich Gauss (1777–1855) was at school he was always quick to answer mathematics questions. One day his teacher, hoping for half an hour of peace and quiet, told his class to add up all the whole numbers from 1 to 100. Almost at once the 10-year-old Gauss announced that he had done it and that the answer was 5050.

Gauss had not of course added the terms one by one. Instead he wrote the series down twice, once in the given order and once backwards, and added the two together:

$$\begin{aligned} S &= 1 + 2 + 3 + \ldots + 98 + 99 + 100 \\ S &= 100 + 99 + 98 + \ldots + 3 + 2 + 1. \end{aligned}$$

Adding, $2S = 101 + 101 + 101 + \ldots + 101 + 101 + 101.$

Since there are 100 terms in the series,

$$\begin{aligned} 2S &= 101 \times 100 \\ S &= 5050. \end{aligned}$$

The numbers 1, 2, 3, … , 100 form an arithmetic sequence with common difference 1. Gauss' method can be used for finding the sum of any arithmetic series.

It is common to use the letter S to denote the sum of a series. When there is any doubt as to the number of terms that are being summed, this is indicated by a subscript: S_5 indicates five terms, S_n indicates n terms.

Worked example

Find the value of $6 + 4 + 2 + \ldots + (-32)$.

Solution

This is an arithmetic progression, with common difference –2. The number of terms, n, can be calculated using

$$n = \frac{l-a}{d} + 1$$

$$n = \frac{-32-6}{-2} + 1$$

$$= 20$$

The sum S of the progression is then found as follows:

$$\begin{array}{rcl} S &=& 6 + 4 + \ldots - 30 - 32 \\ S &=& -32 + (-30) - \ldots + 4 + 6 \\ \hline 2S &=& -26 + (-26) + \ldots + (-26) + (-26). \end{array}$$

Since there are 20 terms, this gives $2S = -26 \times 20$, so $S = -26 \times 10 = -260$.

Generalising this method by writing the series in the conventional notation gives:

$$\begin{array}{rcl} S_n &=& [a] + [a+d] + \ldots + [a+(n-2)d] + [a+(n-1)d] \\ S_n &=& [a+(n-1)d] + [a+(n-2)d] + \ldots + [a+d] + [a] \\ \hline 2S_n &=& [2a+(n-1)d] + [2a+(n-1)d] + \ldots + [2a+(n-1)d] + [2a+(n-1)d] \end{array}$$

Since there are n terms, it follows that

$$S_n = \tfrac{1}{2}n[2a+(n-1)d].$$

This result can also be written as

$$S_n = \tfrac{1}{2}n(a+l).$$

Worked example

Find the sum of the first 100 terms of the progression

$$3\tfrac{1}{3},\ 3\tfrac{2}{3}, 4,\ \ldots$$

Solution

In this arithmetic progression

$$a = 3\tfrac{1}{3}, d = \tfrac{1}{3} \text{ and } n = 100.$$

Using $S_n = \frac{1}{2}n[2a+(n-1)d]$

$$S_n = \tfrac{1}{2} \times 100\left(6\tfrac{2}{3} + 99 \times \tfrac{1}{3}\right)$$

$$= 1983\tfrac{1}{3}.$$

Worked example

Tatjana starts a part-time job on a salary of \$10000 per year, and this increases by \$500 each year. Assuming that, apart from the annual increment, Tatjana's salary does not increase, find

a her salary in the 5th year

b the length of time she has been working to receive total earnings of \$122500.

Solution

Tatjana's annual salaries (in dollars) form the arithmetic sequence

$$10000, 10500, 11000, \ldots$$

with first term $a = 10000$, and common difference $d = 500$.

a Her salary in the 5th year is calculated using:

$$\begin{aligned} & u_k = a + (k-1)d \\ \Rightarrow \quad & u_5 = 10000 + (5-1) \times 500 \\ & = 12000. \end{aligned}$$

b The number of years that have elapsed when her total earnings are \$122500 is given by:

$$S = \tfrac{1}{2}n[2a + (n-1)d]$$

where $S = 122500$, $a = 10000$ and $d = 500$.

This gives $\quad 122\,500 = \tfrac{1}{2}n[2 \times 10000 + 500(n-1)]$.

This simplifies to the quadratic equation:

$$n^2 + 39n - 490 = 0.$$

Factorising,

$$\begin{aligned} & (n-10)(n+49) = 0 \\ \Rightarrow \; & n = 10 \text{ or } n = -49. \end{aligned}$$

The root $n = -49$ is irrelevant, so the answer is $n = 10$.

Tatjana has earned a total of \$122 500 after 10 years.

Exercise 12.2

1 Are the following sequences arithmetic?
If so, state the common difference and the seventh term.

a 28, 30, 32, 34, …
b 1, 1, 2, 3, 5, 8, …
c 3, 9, 27, 81, …
d 5, 9, 13, 17, …
e 12, 8, 4, 0, …

2 The first term of an arithmetic sequence is –7 and the common difference is 4.

a Find the eighth term of the sequence.
b The last term of the sequence is 65. How many terms are there in the sequence?

3 The first term of an arithmetic sequence is 10, the seventh term is 46 and the last term is 100.
 a Find the common difference.
 b Find how many terms there are in the sequence.

4 There are 30 terms in an arithmetic progression.
The first term is −4 and the last term is 141.
 a Find the common difference.
 b Find the sum of the terms in the progression.

5 The kth term of an arithmetic progression is given by

$u_k = 12 + 4k$.

 a Write down the first three terms of the progression.
 b Calculate the sum of the first 12 terms of this progression.

6 Below is an arithmetic progression.

$118 + 112 + \ldots + 34$

 a How many terms are there in the progression?
 b What is the sum of the terms in the progression?

7 The fifth term of an arithmetic progression is 32 and the tenth term is 62.
 a Find the first term and the common difference.
 b The sum of all the terms in this progression is 350.
How many terms are there?

8 The ninth term of an arithmetic progression is three times the second term, and the first term is 5. The sequence has 20 terms.
 a Find the common difference.
 b Find the sum of all the terms in the progression.

9 **a** Find the sum of all the odd numbers between 150 and 250.
 b Find the sum of all the even numbers from 150 to 250 inclusive.
 c Find the sum of the terms of the arithmetic sequence with first term 150, common difference 1 and 101 terms.
 d Explain the relationship between your answers to parts **a**, **b** and **c**.

10 The first term of an arithmetic progression is 9000 and the tenth term is 3600.
 a Find the sum of the first 20 terms of the progression.
 b After how many terms does the sum of the progression become negative?

11 An arithmetic progression has first term −2 and common difference 7.
 a Write down a formula for the nth term of the progression. Which term of the progression equals 110?
 b Write down a formula for the sum of the first n terms of the progression. How many terms of the progression are required to give a sum equal to 2050?

12 Luca's starting salary in a company is \$45 000. During the time he stays with the company, it increases by \$1800 each year.
 a What is his salary in his sixth year?
 b How many years has Luca been working for the company when his total earnings for all his years there are \$531 000?

Exercise 12.2 (cont)

13 A jogger is training for a 5 km charity run. He starts with a run of 400 m, then increases the distance he runs in training by 100 m each day.

a How many days does it take the jogger to reach a distance of 5 km in training?

b What total distance will he have run in training by then?

14 A piece of string 20 m long is to be cut into pieces such that the lengths of the pieces form an arithmetic sequence.

a If the lengths of the longest and shortest pieces are 2 m and 50 cm respectively, how many pieces are there?

b If the length of the longest piece is 185 cm, how long is the shortest piece?

15 The ninth term of an arithmetic progression is 95 and the sum of the first four terms is –10.

a Find the first term of the progression and the common difference.

The nth term of the progression is 200.

b Find the value of n.

16 Following knee surgery, Adankwo has to do squats as part of her physiotherapy programme. Each day she must do 4 more squats than the day before. On the eighth day she did 31 squats. Calculate how many squats Adankwo completed:

a on the first day

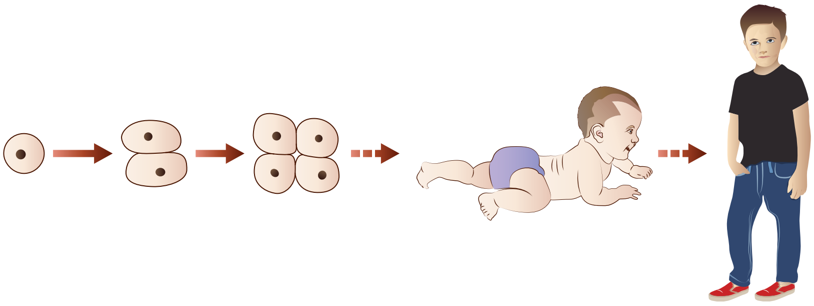
b in total by the end of the seventh day

c in total by the end of the nth day

d in total from the end of the nth day to the end of the $(2n)$th day.

Simplify your answer.

Geometric progressions



A human being begins life as one cell, which divides into two, then four…

The terms of a **geometric sequence** or **geometric progression (G.P.)** are formed by multiplying one term by a fixed number, the **common ratio**, to obtain the next. This can be written inductively as:

$$u_{k+1} = ru_k \text{ with first term } u_1.$$

The sum of the terms of a geometric sequence is called a **geometric series**.

Notation

The following conventions are used in this book to describe geometric progressions:

- first term $u_1 = a$
- common ratio $= r$
- number of terms $= n$
- the general term, u_k, is that in position k (i.e. the kth term).

Thus in the geometric progression 2, 6, 18, 54, 162

$a = 2$, $r = 3$ and $n = 5$.

The terms of this sequence are formed as follows:

$$\begin{aligned}
u_1 &= a & &= 2 & & \\
u_2 &= a \times r & &= 2 \times 3 & &= 6 \\
u_3 &= a \times r^2 & &= 2 \times 3^2 & &= 18 \\
u_4 &= a \times r^3 & &= 2 \times 3^3 & &= 54 \\
u_5 &= a \times r^4 & &= 2 \times 3^4 & &= 162.
\end{aligned}$$

This shows that in each case the power of r is one less than the number of the term: $u_5 = ar^4$ and 4 is one less than 5. This can be written deductively as

$$u_k = ar^{k-1}.$$

For the last term this becomes

$$u_n = ar^{n-1}.$$

These are both general formulae so apply to any geometric sequence.

Given two consecutive terms of a geometric sequence, you can always find the common ratio by dividing the later term by the earlier term. For example, the geometric sequence … 7, 9, … has common ratio $r = \frac{9}{7}$.

Worked example

Find the ninth term in the geometric sequence 7, 28, 112, 448, …

Solution

In the sequence, the first term $a = 7$ and the common ratio $r = 4$.
Using $u_k = ar^{k-1}$

$$\begin{aligned}
u_9 &= 7 \times 4^8 \\
&= 458\,752.
\end{aligned}$$

Worked example

How many terms are there in the geometric sequence 3, 15, 75, … , 29 296 875?

Solution

Since it is a geometric sequence and the first two terms are 3 and 15, you can immediately write down

First term: $a = 3$
Common ratio: $r = 5$

The third term allows you to check you are right.

$$15 \times 5 = 75 \quad ✓$$

The nth term of a geometric sequence is ar^{n-1}, so in this case

$$3 \times 5^{n-1} = 29\,296\,875$$

Dividing by 3 gives

$$5^{n-1} = 9\,765\,625$$

Alternatively, you could find the solution by using trial and improvement and a calculator, since you know n must be a whole number.

Using logarithms, $\lg(5)^{(n-1)} = \lg 9765625$

$$\Rightarrow \quad (n-1)\lg 5 = \lg 9765625$$

$$\Rightarrow \quad n - 1 = \frac{\lg 9765625}{lg 5} = 10$$

So $n = 11$ and there are 11 terms in the sequence.

Discussion point

How would you use a spreadsheet to solve the equation $5^{n-1} = 9765625$?

The sum of the terms of a geometric progression

This chapter began with the story of Sissa ben Dahir's reward for inventing chess. In the discussion point on page 184, you were asked how much grain would have been placed on the last square. This situation also gives rise to another question:

How many grains of wheat was the inventor actually asking for?

The answer is the geometric series with 64 terms and common ratio 2:

$$1 + 2 + 4 + 8 + \ldots + 2^{63}.$$

This can be summed as follows.

Call the series S:

$$S = 1 + 2 + 4 + 8 + \ldots + 2^{63}. \quad ①$$

Now multiply it by the common ratio, 2:

$$2S = 2 + 4 + 8 + 16 + \ldots + 2^{64}. \quad ②$$

Then subtract ① from ②:

$$\begin{aligned} ② \quad 2S &= \phantom{1+{}} 2 + 4 + 8 + 16 + \ldots + 2^{63} + 2^{64} \\ ① \quad S &= 1 + 2 + 4 + 8 \phantom{{}+16} + \ldots + 2^{63} \\ \text{Subtracting:} \quad S &= -1 + 0 + 0 + 0 + \ldots + 2^{64}. \end{aligned}$$

The total number of wheat grains requested was therefore $2^{64} - 1$ (which is about 1.85×10^{19}).

Discussion point

How many tonnes of wheat is this, and how many tonnes would you expect there to be in China at any time?

(One hundred grains of wheat weigh about 2 grams. The world annual production of all cereals is about 1.8×10^9 tonnes.)

Note

The method shown above can be used to sum any geometric progression.

Worked example

Find the sum of $0.04 + 0.2 + 1 + \ldots + 78\,125$.

Solution

This is a geometric progression with common ratio 5.

Let $S = 0.04 + 0.2 + 1 + \ldots + 78\,125.$ ①

Multiplying by the common ratio, 5, gives:

$5S = 0.2 + 1 + 5 + \ldots + 78\,125 + 390\,625.$ ②

Subtracting ① from ②:

$$\begin{aligned} 5S &= \phantom{0.04 +{}} 0.2 + 1 + 5 + \ldots + 78\,125 + 390\,625 \\ S &= 0.04 + 0.2 + 1 + 5 + \ldots + 78\,125 \\ \hline 4S &= -0.04 + 0 + \ldots + 0 + 390\,625 \end{aligned}$$

This gives $4S = 390\,624.96$

$\Rightarrow \quad S = 97\,656.24$

The same method can be applied to the general geometric progression to give a formula for its value:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}. \qquad ①$$

Multiplying by the common ratio, r, gives:

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n. \qquad ②$$

Subtracting ① from ②, as before, gives:

$$rS_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n - 1)$$

so $$S_n = \frac{a(r^n - 1)}{(r-1)}.$$

This can also be written as:

$$S_n = \frac{a(1-r^n)}{(1-r)}.$$

Worked example

a Solve the simultaneous equations $ar^2 = 6$
$ar^4 = 54$

b Find in each case the sum of the first five terms of the geometric progression.

Solution

a $ar^2 = 6 \Rightarrow a = \dfrac{6}{r^2}$

Substituting into $ar^4 = 54$ gives $\dfrac{6}{r^2} \times r^4 = 54$

$\Rightarrow r^2 = 9$

$\Rightarrow \quad r = \pm 3$

Substituting in $ar^2 = 6$ gives $a = \frac{2}{3}$ in both cases.

b When $r = +3$ terms are $\frac{2}{3}$, 2, 6, 18, 54 Sum = $80\frac{2}{3}$

When $r = -3$ terms are $\frac{2}{3}$, −2, 6, −18, 54 Sum = $40\frac{2}{3}$

Infinite geometric progressions

The progression $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is geometric, with common ratio $\frac{1}{2}$.

Summing the terms one by one gives $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16} \ldots$

Clearly the more terms you add, the nearer the sum gets to 2. In the limit, as the number of terms tends to infinity, the sum tends to 2.

As $n \to \infty, S_n \to 2$.

This is an example of a **convergent** series. The sum to infinity is a finite number.

You can see this by substituting $a = 1$ and $r = \frac{1}{2}$ in the formula for the sum of the series:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

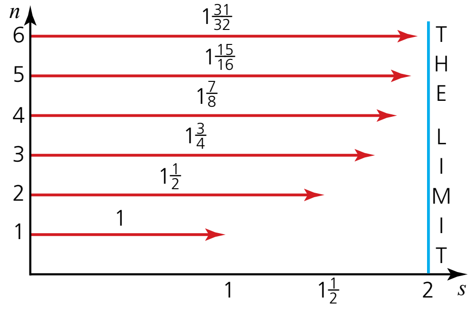
giving

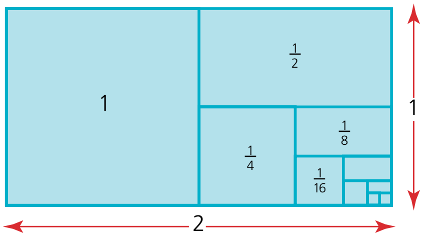
$$S_n = \frac{1 \times \left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \frac{1}{2}\right)}$$

$$= 2 \times \left(1 - \left(\tfrac{1}{2}\right)^n\right).$$

The larger the number of terms, n, the smaller $\left(\frac{1}{2}\right)^n$ becomes and so the nearer S_n is to the limiting value of 2, as shown on the left. Notice that $\left(\frac{1}{2}\right)^n$ can never be negative, however large n becomes; so S_n can never exceed 2.

Notice how representing all of the terms of the geometric progression as in these diagrams shows that the sum can never exceed 2.





In the general geometric series $a + ar + ar^2 + \ldots$ the terms become progressively smaller in size if the common ratio r is between −1 and 1. In such cases, the geometric series is **convergent**.

If, on the other hand, the value of r is greater than 1 (or less than −1), the terms in the series become larger and larger in size and so the series is described as **divergent**.

A series corresponding to a value of r of exactly +1 consists of the first term a repeated over and over again. A sequence corresponding to a value of r of exactly −1 oscillates between $+a$ and $-a$. Neither of these is convergent.

It only makes sense to talk about the sum of an infinite series if it is convergent. Otherwise the sum is undefined.

The condition for a geometric series to converge, $-1 < r < 1$, ensures that as $n \to \infty$, $r^n \to 0$, and so the formula for the sum of a geometric series:

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

can be rewritten for an infinite series as:

$$S_\infty = \frac{a}{1-r}.$$

Worked example

Find the sum of the terms of the infinite progression 0.4, 0.04, 0.004, …

Solution

This is a geometric progression with $a = 0.4$ and $r = 0.1$.
Its sum is given by:

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{0.4}{1-0.1} \\ &= \frac{0.4}{0.9} \\ &= \frac{4}{9}. \end{aligned}$$

Note

You may have noticed that the sum of the series $0.4 + 0.04 + 0.004 + \ldots$ is $0.\dot{4}$, and that this recurring decimal is the same as $\frac{4}{9}$.

Worked example

The first three terms of an infinite geometric progression are 75, 45 and 27.

a Write down the common ratio.

b Find the sum of the terms of the progression.

Solution

a The common ratio is $\frac{45}{75} = \frac{3}{5}$.

using $S_\infty = \frac{a}{1-r}$ → **b** $S_\infty = \dfrac{75}{1-\frac{3}{5}} = 187.5$

Discussion point

A paradox

Consider the following arguments.

i $S = 1 - 2 + 4 - 8 + 16 - 32 + 64 - \dots$

$\Rightarrow S = 1 - 2(1 - 2 + 4 - 8 + 16 - 32 + \dots)$

$= 1 - 2S$

$\Rightarrow 3S = 1$

$\Rightarrow S = \frac{1}{3}$.

ii $S = 1 + (-2 + 4) + (-8 + 16) + (-32 + 64) + \dots$

$\Rightarrow S = 1 + 2 + 8 + 32 + \dots$

So S diverges towards $+\infty$.

iii $S = (1 - 2) + (4 - 8) + (16 - 32) + \dots$

$\Rightarrow S = -1 - 4 - 8 - 16 \dots$

So S diverges towards $-\infty$.

What is the sum of the series: $\frac{1}{3}$, $+\infty$, $-\infty$, or something else?

Exercise 12.3

1 Are the following sequences geometric?
If so, state the common ratio and calculate the seventh term.
- a 3, 6, 12, 24, …
- b 3, 6, 9, 12, …
- c 10, –10, 10, –10, 10, …
- d 1, 1, 1, 1, 1, 1, …
- e 15, 10, 5, 0, –5, …
- f $10,\ 5,\ \frac{5}{2},\ \frac{5}{4},\ \frac{5}{8},\ \dots$
- g 2, 2.2, 2.22, 2.222, …

2 A geometric sequence has first term 5 and common ratio 2.
The sequence has seven terms.
- a Find the last term.
- b Find the sum of the terms in the sequence.

3 The first term of a geometric sequence of positive terms is 3 and the fifth term is 768.
- a Find the common ratio of the sequence.
- b Find the eighth term of the sequence.

4 A geometric sequence has first term $\frac{1}{16}$ and common ratio 4.
- a Find the fifth term.
- b Which is the first term of the sequence that exceeds 1000?

5
- a Find how many terms there are in the following geometric sequence: 7, 14, …, 3584.
- b Find the sum of the terms in this sequence.

Exercise 12.3 (cont)

6 **a** Find how many terms there are in the following geometric sequence: 100, 50, …, 0.390 625.
b Find the sum of the terms in this sequence.

7 The fourth term of a geometric progression is 36 and the eighth term is 576. All the terms are positive.
a Find the common ratio.
b Find the first term.
c Find the sum of the first ten terms.

8 The first three terms of an infinite geometric progression are 8, 4 and 2.
a State the common ratio of this progression.
b Calculate the sum to infinity of its terms.

9 The first three terms of an infinite geometric progression are 0.8, 0.08 and 0.008.
a Write down the common ratio for this progression.
b Find, as a fraction, the sum to infinity of the terms of this progression.
c Find the sum to infinity of the geometric progression

$$0.8 - 0.08 + 0.008 - \ldots$$

and hence show that $\frac{8}{11} = 0.\dot{7}\dot{2}$.

10 The first three terms of a geometric sequence are 100, 70 and 49.
a Write down the common ratio of the sequence.
b Which is the position of the first term in the sequence that has a value less than 1?
c Find the sum to infinity of the terms of this sequence.
d After how many terms is the sum of the sequence greater than 99% of the sum to infinity?

11 A geometric progression has first term 10 and its sum to infinity is 15.
a Find the common ratio.
b Find the sum to infinity if the first term is excluded from the progression.

12 The first four terms in an infinite geometric series are 216, 72, 24, 8.
a What is the common ratio r?
b Write down an expression for the nth term of the series.
c Find the sum of the first n terms of the series.
d Find the sum to infinity.
e How many terms are needed for the sum to be greater than 323.999?

13 A tank is filled with 10 litres of water. Half the water is removed and replaced with anti-freeze and then thoroughly mixed. Half this mixture is then removed and replaced with anti-freeze. The process continues.
a Find the first five terms in the sequence of amounts of water in the tank at each stage.
b Find the first five terms in the sequence of amounts of anti-freeze in the tank at each stage.
c Is either of these sequences geometric? Explain.

14 A pendulum is set swinging. Its first oscillation is through an angle of 20°, and each following oscillation is through 95% of the angle of the one before it.

a After how many swings is the angle through which it swings less than 1°?

b What is the total angle it has swung through at the end of its tenth oscillation?

15 A ball is thrown vertically upwards from the ground. It rises to a height of 15 m and then falls and bounces. After each bounce it rises vertically to $\frac{5}{8}$ of the height from which it fell.

a Find the height to which the ball bounces after the nth impact with the ground.

b Find the total distance travelled by the ball from the first throw to the tenth impact with the ground.

16 The first three terms of an arithmetic sequence, a, $a + d$ and $a + 2d$, are the same as the first three terms, a, ar, ar^2, of a geometric sequence ($a \neq 0$). Show that this is only possible if $r = 1$ and $d = 0$.

17 a, b and c are three consecutive terms in a sequence.

a Prove that if the sequence is an arithmetic progression then $a + c = 2b$.

b Prove that if the sequence is a geometric progression then $ac = b^2$.

18 a Solve the simultaneous equations $ar = 12$, $ar^5 = 3072$ (there are two possible answers).

b In each case, find the sum of the first ten terms of the geometric progression with first term a and common ratio r.

Past-paper questions

1 Find the values of the positive constants p and q such that, in the binomial expansion of $(p + qx)^{10}$, the coefficient of x^5 is 252 and the coefficient of x^3 is 6 times the coefficient of x^2. [8]

Cambridge O Level Additional Mathematics 4037
Paper 11 Q9 June 2012
Cambridge IGCSE Additional Mathematics 0606
Paper 11 Q9 June 2012

2 (i) Find the coefficient of x^3 in the expansion of $\left(1 - \frac{x}{2}\right)^{12}$. [2]

(ii) Find the coefficient of x^3 in the expansion of $(1 + 4x)\left(1 - \frac{x}{2}\right)^{12}$. [3]

Cambridge O Level Additional Mathematics 4037
Paper 21 Q2 June 2011
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q2 June 2011

3 **(i)** Find the first four terms in the expansion of $(2+x)^6$ in ascending powers of x. [3]

(ii) Hence find the coefficient of x^3 in the expansion of $(1+3x)(1-x)(2+x)^6$. [4]

Cambridge O Level Additional Mathematics 4037
Paper 21 Q7 June 2013
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q7 June 2013

Learning outcomes

Now you should be able to:

★ use the binomial theorem for expansion of $(a+b)^n$ for positive integer n

★ use the general term $\binom{n}{r}a^{n-r}b^r$, $0 \leqslant r \leqslant n$ (knowledge of the greatest term and properties of the coefficients is not required)

★ recognise arithmetic and geometric progressions

★ use the formulae for the nth term and for the sum of the first n terms to solve problems involving arithmetic or geometric progressions

★ use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression.

Key points

✔ An expression of the form $(ax+b)^n$ where n is an integer is called a **binomial expression**.

✔ **Binomial coefficients**, denoted by $\binom{n}{r}$ or ${}^nC_r$ can be found:
- using Pascal's triangle
- using tables
- using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

✔ The binomial expansion of $(1+x)^n$ can also be written as

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots nx^{n-1} + x^n$$

✔ A sequence is an ordered set of numbers, $u_1, u_2, u_3, \ldots, u_k, \ldots u_n$, where u_k is the general term.

✔ In an arithmetic sequence, $u_{k+1} = u_k + d$ where d is a fixed number called the **common difference**.

✔ In a geometric sequence, $u_{k+1} = ru_k$ where r is a fixed number called the **common ratio**.

✔ For an arithmetic progression with first term a, common difference d and n terms

- the kth term $u_k = a + (k - 1)d$
- the last term $l = a + (n - 1)d$
- the sum of the terms $= \frac{1}{2}n(a + l) = \frac{1}{2}n[2a + (n - 1)d]$

✔ For a geometric progression with first term a, common ratio r and n terms

- the kth term $a_k = ar^{k-1}$
- the last term $a_n = ar^{n-1}$
- the sum of the terms $= \frac{a(r^n - 1)}{(r - 1)}$ for $r > 1$ or $\frac{a(1 - r^n)}{(1 - r)}$ for $r < 1$

✔ For an infinite geometric series to converge, $-1 < r < 1$. In this case the sum of all terms is given by $\frac{a}{1 - r}$.