11 Permutations and combinations

It always seems impossible until it is done.

Nelson Mandela (1918 – 2013)

Discussion point

The combination lock has four numbers to be found and six choices for each number: 1, 2, 3, 4, 5 or 6. Suppose you have no idea what the code is, but you need to open the lock. It may seem like an impossible situation initially, but what if you try every possible combination of numbers systematically? How many possible combinations are there? Estimate how long it will take you to open the lock.

Factorials

Worked example

Winni is tidying her bookshelf and wants to put her five maths books together. In how many different ways can she arrange them?

Solution

There are 5 possible books that can go in the first space on the shelf.

There are 4 possible books for the second space.

There are 3 for the third space, 2 for the fourth and only 1 book left for the fifth space.

The total number of arrangements is therefore

5	$\times$	4	$\times$	3	$\times$	2	$\times$	1	= 120
Book 1		Book 2		Book 3		Book 4		Book 5	

This number, $5 \times 4 \times 3 \times 2 \times 1$, is called **5 factorial** and is written 5!

n must be a positive integer.

This example illustrates a general result. The number of ways of placing n different objects in a line is $n!$, where $n! = n \times (n-1) \times (n-2) \ldots \times 3 \times 2 \times 1$.

By convention, a special case is made for $n = 0$. The value of 0! is taken to be 1.

Worked example

Find the value of each of the following:

a 2! **b** 3! **c** 4! **d** 5! **e** 10!

Solution

a $2! = 2 \times 1 = 2$

b $3! = 3 \times 2 \times 1 = 6$

c $4! = 4 \times 3 \times 2 \times 1 = 24$

d $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

e $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3\,628\,800$

You can see that factorials go up very quickly in size.

Worked example

a Calculate $\frac{7!}{5!}$

b Calculate $\frac{5! \times 4! \times 3!}{6! \times 2!}$

Solution

a $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ and $5! = 5 \times 4 \times 3 \times 2 \times 1$

So $\frac{7!}{5!} = \frac{7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 42$

This result can be generalised as $\frac{n!}{m!} = \frac{n \times (n-1) \times (n-2) \times \ldots \times (m+1) \times m!}{m!}$ $= n \times (n-1) \times (n-2) \times \ldots \times (m+1)$

You can also write 7! as $7 \times 6 \times 5!$

n > m

Using this, $\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$

b $\frac{5 \times 4 \times 3 \times 2 \times 1 \ \times\ 4 \times 3 \times 2 \times 1 \ \times\ 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \ \times\ 2 \times 1} = 12$

Worked example

a Find the number of ways in which all six letters in the word FOURTH can be arranged.

b In how many of these arrangements are the letters O and U next to each other?

Solution

a There are six choices for the first letter (F, O, U, R, T, H). Then there are five choices for the next letter, then four for the fourth letter and so on. So the number of arrangements of the letters is

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$

b The O and the U are to be together, so you can treat them as a single letter.

So there are five choices for the first letter '(F, OU, R, T or H)', four choices for the next letter and so on.

So the number of arrangements of these five 'letters' is

$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

However $\boxed{\text{OU}}\ \boxed{\text{F}}\ \boxed{\text{R}}\ \boxed{\text{T}}\ \boxed{\text{H}}$

is different from $\boxed{\text{UO}}\ \boxed{\text{F}}\ \boxed{\text{R}}\ \boxed{\text{T}}\ \boxed{\text{H}}$

So each of the 120 arrangements can be arranged into two different orders.

The total number of arrangements with the O and U next to each other is

$2 \times 5! = 240$

Notice that the total number of ways of arranging the letters with the U and the O apart is $720 - 240 = 480$

Exercise 11.1

1 Calculate: **a** $7!$ **b** $\frac{9!}{7!}$ **c** $\frac{4! \times 6!}{7! \times 2!}$

2 Simplify: **a** $\frac{n!}{(n+1)!}$ **b** $\frac{(n-2)!}{(n-3)!}$

3 Simplify: **a** $\frac{(n+2)!}{n!}$ **b** $\frac{(n+1)!}{(n-1)!}$

4 Write in factorial notation:

a $\frac{9 \times 8 \times 7}{6 \times 5 \times 4}$ **b** $\frac{14 \times 15}{5 \times 4 \times 3 \times 2}$ **c** $\frac{(n+2)(n+1)n}{4 \times 3 \times 2}$

5 Factorise: **a** $6! + 7!$ **b** $n! + (n-1)!$

6 Write the number 42 using factorials only.

7 How many different four-letter arrangements can be formed from the letters P, Q, R and S if letters cannot be repeated?

8 How many different ways can seven books be arranged in a row on a shelf?

9 There are five drivers in a motoring rally.

How many different ways are there for the five drivers to finish?

Exercise 11.1 (cont)

10 There are five runners in a 60-metre hurdles race, one from each of the nations Japan, South Korea, Cambodia, Malaysia and Thailand.

How many different finishing orders are there?

11 Toben listens to 15 songs from a playlist. If he selects 'shuffle' so the songs are played in a random order, in how many different orders could the songs be played?

12 How many different arrangements are there of the letters in each word?

a ASK **b** QUESTION **c** SINGAPORE
d GOVERN **e** VIETNAM **f** MAJORITY

13 How many arrangements of the letters in the word ARGUMENT are there if:
a there are no restrictions on the order of the letters
b the first letter is an A
c the letters A and R must be next to each other
d the letters G and M must not be next to each other.

Permutations

In some situations, such as a race, the finishing order matters. An ordered arrangement of a number of people, objects or operations is called a **permutation**.

Worked example


Joyeeta

I should be one of the judges! When I saw the 10 contestants in the cookery competition, I knew which ones I thought were the best three. Last night they announced the results and I had picked the same three contestants in the same order as the judges!

What is the probability of Joyeeta's result?

Solution

The winner can be chosen in 10 ways.

The second contestant can be chosen in 9 ways.

The third contestant can be chosen in 8 ways.

Thus the total number of ways of placing three contestants in the first three positions is $10 \times 9 \times 8 = 720$. So the probability that Joyeeta's selection is correct is $\frac{1}{720}$.

In this example attention is given to the order in which the contestants are placed. The solution required a **permutation** of three objects from ten.

In general the number of permutations, ${}^nP_r$, of r objects from n is given by

$${}^nP_r = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1).$$

This can be written more compactly as

$${}^nP_r = \frac{n!}{(n-r)!}$$

Worked example

Five people go to the theatre. They sit in a row with eight seats. Find how many ways can this be done if:

a they can sit anywhere

b all the empty seats are next to each other.

Solution

a The first person to sit down has a choice of eight seats.

The second person to sit down has a choice of seven seats.

The third person to sit down has a choice of six seats.

The fourth person to sit down has a choice of five seats.

The fifth person to sit down has a choice of four seats.

So the total number of arrangements is $8 \times 7 \times 6 \times 5 \times 4 = 6720$.

This is a permutation of five objects from eight, so a quicker way to work this out is:

number of arrangements $= {}^8P_5 = 6720$.

b Since all three empty seats are to be together you can consider them to be a single 'empty seat', albeit a large one!

So there are six seats to seat five people.

So the number of arrangements is ${}^6P_5 = 720$.

Combinations

In other situations, order is not important, for example, choosing five of eight students to go to the theatre. You are not concerned with the order in which people or objects are chosen, only with which ones are picked. A selection where order is not important is called a **combination**.

A maths teacher is playing a game with her students. Each student selects six numbers out of a possible 19 (numbers $1, 2, \ldots, 19$). The maths teacher then uses a random number machine to generate six numbers. If a student's numbers match the teacher's numbers then they win a prize.

Discussion point

You have the six winning numbers. Does it matter in which order the machine picked them?

The teacher says that the probability of an individual student picking the winning numbers is about 1 in 27 000. How can you work out this figure?

The key question is, how many ways are there of choosing six numbers out of of 19?

If the order mattered, the answer would be ${}^{19}P_6$, or $19 \times 18 \times 17 \times 16 \times 15 \times 14$.

However, the order does not matter. The selection 1, 3, 15, 19, 5 and 18 is the same as 15, 19, 1, 5, 3, 18 and as 18, 1, 19, 15, 3, 5, and lots more. For each set of six numbers there are 6! arrangements that all count as being the same.

So, the number of ways of selecting six numbers, given that the order does not matter, is

$$\frac{19 \times 18 \times 17 \times 16 \times 15 \times 14}{6!}.$$

This is $\frac{{}^{19}P_6}{6!}$

This is called the number of **combinations** of 6 objects from 19 and is denoted by ${}^{19}C_6$.

Discussion point

Show that ${}^{19}C_6$ can be written as $\frac{19!}{6!\,13!}$.

Returning to the maths teacher's game, it follows that the probability of a student winning is $\frac{1}{{}^{19}C_6}$.

This is about 27 000.

Discussion point

How does the probability change if there are 29, 39 and 49 numbers to choose from?

This example shows a general result, that the number of ways of selecting r objects from n, when the order does not matter, is given by

$${}^{n}C_r = \frac{n!}{r!(n-r)!} = \frac{{}^{n}P_r}{r!}$$

Discussion point

How can you prove this general result?

Another common notation for ${}^nC_r$ is $\binom{n}{r}$. Both notations are used in this book to help you become familiar with them.

Caution: The notation $\binom{n}{r}$ looks exactly like a column vector and so there is the possibility of confusing the two. However, the context will usually make the meaning clear.

Worked example

A student representative committee of five people is to be chosen from nine applicants. How many different selections are possible?

Solution

Number of selections $= \binom{9}{5} = \frac{9!}{5! \times 4!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126$

Worked example

In how many ways can a committee of five people be selected from five applicants?

Solution

Common sense tells us that there is only one way to make the committee, that is by appointing all applicants. So ${}^5C_5 = 1$. However, if we work from the formula

$$ {}^5C_5 = \frac{5!}{5!0!} = \frac{1}{0!} $$

for this to equal 1 requires the convention that 0! is taken to be 1.

Discussion point

Use the convention $0! = 1$ to show that ${}^nC_0 = {}^nC_n = 1$ for all values of n.

Exercise 11.2

1 a Find the values of i ${}^7P_3$ ii ${}^9P_4$ iii ${}^{10}P_8$
  b Find the values of i ${}^7C_3$ ii ${}^9C_4$ iii ${}^{10}C_8$
  c Show that, for the values of n and r in parts **a** and **b**, ${}^nC_r = \frac{{}^nP_r}{r!}$.

2 There are 15 competitors in a camel race. How many ways are there of guessing the first three finishers?

3 A group of 6 computer programmers is to be chosen to work the night shift from a set of 14 programmers. In how many ways can the programmers be chosen if the 6 chosen must include the shift-leader who is one of the 14?

Exercise 11.2 (cont)

4 Zaid decides to form a band. He needs a bass player, a guitarist, a keyboard player and a drummer. He invites applications and gets 6 bass players, 8 guitarists, 4 keyboard players and 3 drummers. Assuming each person applies only once, in how many ways can Zaid put the band together?

5 A touring party of cricket players is made up of 6 players from each of India, Pakistan and Sri Lanka and 3 from Bangladesh.
  a How many different selections of 11 players can be made for a team?
  b In one match, it is decided to have 3 players from each of India, Pakistan and Sri Lanka and 2 from Bangladesh. How many different team selections can now be made?

6 A committee of four is to be selected from ten candidates, five men and five women.
  a In how many distinct ways can the committee be chosen?
  b Assuming that each candidate is equally likely to be selected, determine the probabilities that the chosen committee contains:
    i no women
    ii two men and two women.

7 A committee of four is to be selected from four boys and six girls. The members are selected at random.
  a How many different selections are possible?
  b What is the probability that the committee will be made up of:
    i all girls
    ii more boys than girls?

8 A factory advertises six positions. Nine men and seven women apply.
  a How many different selections are possible?
  b How many of these include equal numbers of men and women?
  c How many of the selections include no men?
  d How many of the selections include no women?

9 A small business has 14 staff; 6 men and 8 women. The business is struggling and needs to make four members of staff redundant.
  a How many different selections are possible if the four staff are chosen at random?
  b How many different selections are possible if equal numbers of men and women are chosen?
  c How many different selections are possible if there are equal numbers of men and women remaining after the redundancies?

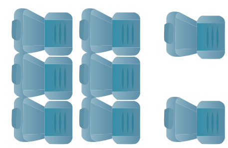
10 A football team consists of a goalkeeper, two defense players, four midfield players and four forwards. Three players are chosen to collect a medal at the closing ceremony of a competition.

How many selections are possible if one midfield player, one defense player and one forward must be chosen?

11 Find how many different numbers can be made by arranging all nine digits of the number 335 688 999 if:
  i there are no restrictions
  ii the number made is a multiple of 5.

12 Nimish is going to install 5 new game apps on her phone. She has shortlisted 2 word games, 5 quizzes and 16 saga games. Nimish wants to have at least one of each type of game. How many different selections of apps could Nimish possibly choose?

13 A MPV has seven passenger seats – one in the front, and three in each of the other two rows.



a In how many ways can all 8 seats be filled from a party of 12 people, assuming that they can all drive?

b In a party of 12 people, 3 are qualified drivers. They hire an MPV and a four-seater saloon car. In how many ways can the party fill the MPV given that one of the drivers must drive each vehicle?

14 Iram has 12 different DVDs of which 7 are films, 3 are music videos and 2 are documentaries.

a How many different arrangements of all 12 DVDs on a shelf are possible if the music videos are all next to each other?

b Iram makes a selection of 2 films, 2 music videos and 1 documentary. How many possible selections can be made?

15 A string orchestra consists of 15 violins, 8 violas, 7 cellos and 4 double basses. A chamber orchestra consisting of 8 violins, 4 violas, 2 cellos and 2 double basses is to be chosen from the string orchestra.

a In how many different ways can the chamber orchestra be chosen?

b Once the chamber orchestra is chosen, how many seating arrangements are possible if each instrument group has their own set of chairs?

c The violinists work in pairs. How many seating arrangements are possible for the violinists if they must sit with their partner?

16 An office car park has 12 parking spaces in a row. There are 9 cars to be parked.

a How many different arrangements are there for parking the 9 cars and leaving 3 empty spaces?

b How many different arrangements are there if the 3 empty spaces are next to each other?

Past-paper questions

1 A school council of 6 people is to be chosen from a group of 8 students and 6 teachers. Calculate the number of different ways that the council can be selected if

(i) there are no restrictions, [2]

(ii) there must be at least 1 teacher on the council and more students than teachers. [3]

After the council is chosen, a chairperson and a secretary have to be selected from the 6 council members.

(iii) Calculate the number of different ways in which a chairperson and a secretary can be selected. [1]

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2 **(a)** **(i)** Find how many different 4-digit numbers can be formed from the digits 1, 3, 5, 6, 8 and 9 if each digit may be used only once. [1]

(ii) Find how many of these 4-digit numbers are even. [1]

(b) A team of 6 people is to be selected from 8 men and 4 women. Find the number of different teams that can be selected if

(i) there are no restrictions, [1]

(ii) the team contains all 4 women, [1]

(iii) the team contains at least 4 men. [3]

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3 Arrangements containing 5 different letters from the word AMPLITUDE are to be made. Find

(a) **(i)** the number of 5-letter arrangements if there are no restrictions, [1]

(ii) the number of 5-letter arrangements which start with the letter A and end with the letter E. [1]

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(Part question: part b omitted)
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(Part question: part b omitted)

Learning outcomes

Now you should be able to:

★ recognise and distinguish between a permutation case and a combination case
★ recall and use the notation $n!$ (with $0! = 1$), and expressions for permutations and combinations of n items taken r at a time
★ answer simple problems on arrangement and selection.

Key points

- The number of ways of arranging n different objects in a line is $n!$ This is read as n factorial.
- $n! = n \times (n-1) \times (n-2) \ldots \times 3 \times 2 \times 1$ where n is a positive integer.
- By convention, $0! = 1$.
- The number of permutations of r objects from n is ${}^{n}P_{r} = \frac{n!}{(n-r)!}$
- The number of combinations of r objects from n is ${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$
- The order matters for permutations, but not for combinations.