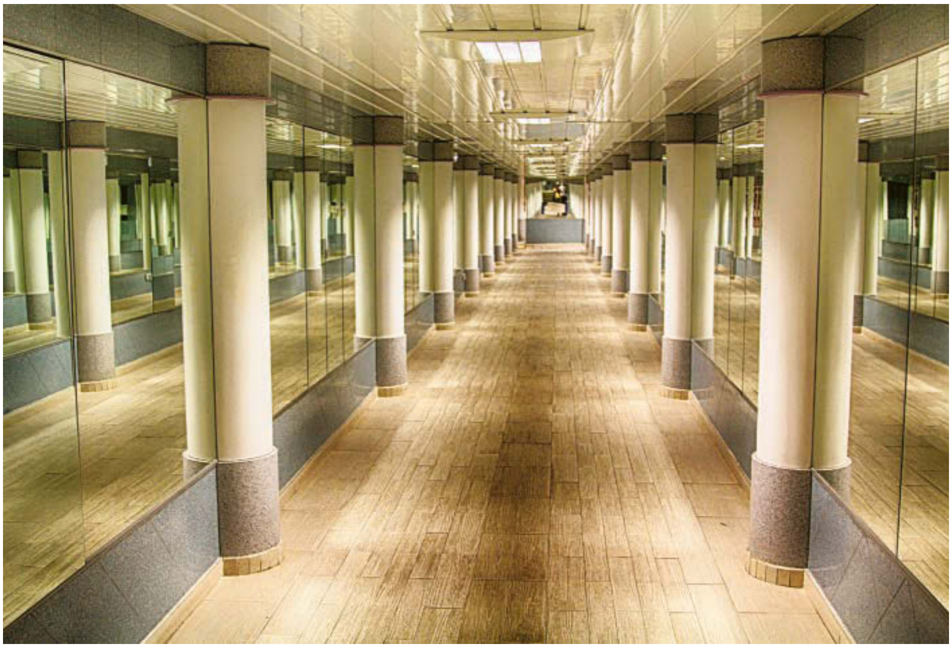
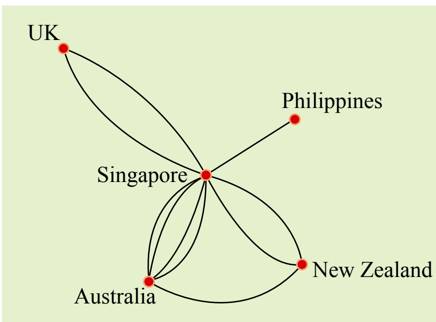
1 Matrices and transformations

> **As for everything else, so for a mathematical theory – beauty can be perceived but not explained.**
> *Arthur Cayley (1821–1895)*





▲ **Figure 1.1** Direct flights between countries by one airline.

Figure 1.1 shows some of the direct flights between countries by one airline. How many direct flights are there from:

- Singapore to Australia
- Australia to New Zealand
- the UK to the Philippines?

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

1.1 Matrices

You can represent the number of direct flights between each pair of countries (shown in Figure 1.1) as an array of numbers like this:

	A	N	P	S	U
A	0	1	0	4	0
N	1	0	0	2	0
P	0	0	0	1	0
S	4	2	1	0	2
U	0	0	0	2	0

The array is called a **matrix** (the plural is **matrices**) and is usually written inside curved brackets.

$$\begin{pmatrix} 0 & 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

It is usual to represent matrices by capital letters, often in bold print.

A matrix consists of rows and columns, and the entries in the various cells are known as **elements**.

The matrix $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$ representing the flights between the counties has 25 elements, arranged in five rows and five columns. **M** is described as a 5×5 matrix, and this is the **order** of the matrix. You state the number of rows first then the number of columns. So, for example, the matrix

$\mathbf{M} = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 0 & 5 \end{pmatrix}$ is a 2×3 matrix and $\mathbf{N} = \begin{pmatrix} 4 & -4 \\ 3 & 4 \\ 0 & -2 \end{pmatrix}$ is a 3×2 matrix.

Special matrices

Some matrices are described by special names that relate to the number of rows and columns or the nature of the elements.

Matrices such as $\begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 5 & 1 \\ 2 & 0 & -4 \\ 1 & 7 & 3 \end{pmatrix}$ that have the same number of rows as columns are called **square matrices**.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 **identity matrix** or **unit matrix**, and similarly $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is called the 3×3 identity matrix. Identity matrices must be square, and are usually denoted by I.

The matrix $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is called the 2×2 **zero matrix**. Zero matrices can be of any order.

Two matrices are said to be **equal** if, and only if, they have the same order and each element in one matrix is equal to the corresponding element in the other matrix. So, for example, the matrices **A** and **D** below are equal, but **B** and **C** are not equal to any of the other matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Working with matrices

Matrices can be added or subtracted if they are of the same order.

$$\begin{pmatrix} 2 & 4 & 0 \\ -1 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 4 \\ 2 & 0 & -5 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 4 \\ 1 & 3 & 0 \end{pmatrix}$$

Add the elements in corresponding positions.

$$\begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 7 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 5 & -1 \end{pmatrix}$$

Subtract the elements in corresponding positions.

But $\begin{pmatrix} 2 & 4 & 0 \\ -1 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$ cannot be evaluated because the matrices are not of the same order. These matrices are **non-conformable** for addition.

You can also multiply a matrix by a **scalar** number:

$$2\begin{pmatrix} 3 & -4 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -8 \\ 0 & 12 \end{pmatrix}$$

Multiply each of the elements by 2.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

Technology note

You can use a calculator to add and subtract matrices of the same order and to multiply a matrix by a number. If you have a calculator that can handle matrices, find out:

- the method for inputting matrices
- how to add and subtract matrices
- how to multiply a matrix by a number for matrices of varying sizes.

Associativity and commutativity

When working with numbers the properties of **associativity** and **commutativity** are often used.

Associativity

Addition of numbers is **associative**.

$$(3 + 5) + 8 = 3 + (5 + 8)$$

When you add numbers, it does not matter how the numbers are grouped, the answer will be the same.

Commutativity

Addition of numbers is **commutative**.

$$4 + 5 = 5 + 4$$

When you add numbers, the order of the numbers can be reversed and the answer will still be the same.

- Give examples to show that subtraction of numbers is not commutative or associative.
- Are matrix addition and matrix subtraction associative and/or commutative?

Exercise 1A

1 Write down the order of these matrices.

(i) $\begin{pmatrix} 2 & 4 \\ 6 & 0 \\ -3 & 7 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 8 & 4 \\ -2 & -3 & 1 \\ 5 & 3 & -2 \end{pmatrix}$ (iii) $\begin{pmatrix} 7 & -3 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ (v) $\begin{pmatrix} 2 & -6 & 4 & 9 \\ 5 & 10 & 11 & -4 \end{pmatrix}$ (vi) $\begin{pmatrix} 8 & 5 \\ -2 & 0 \\ 3 & -9 \end{pmatrix}$

2 For the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 7 & -3 \\ 1 & 4 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 3 & 5 & -9 \\ 2 & 1 & 4 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 0 & -4 & 5 \\ 2 & 1 & 8 \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} -3 & 5 \\ -2 & 7 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

find, where possible

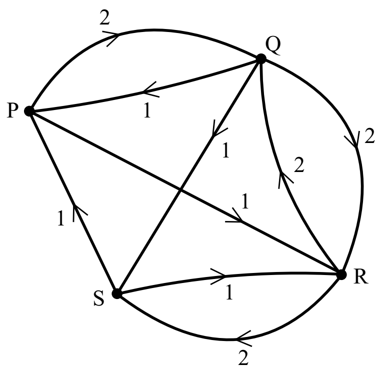
(i) $\mathbf{A} - \mathbf{E}$ (ii) $\mathbf{C} + \mathbf{D}$ (iii) $\mathbf{E} + \mathbf{A} - \mathbf{B}$

(iv) $\mathbf{F} + \mathbf{D}$ (v) $\mathbf{D} - \mathbf{C}$ (vi) $4\mathbf{F}$

(vii) $3\mathbf{C} + 2\mathbf{D}$ (viii) $\mathbf{B} + 2\mathbf{F}$ (ix) $\mathbf{E} - (2\mathbf{B} - \mathbf{A})$

3 The diagram below shows the number of direct ferry crossings on one day offered by a ferry company between cities P, Q, R and S.

The same information is also given in the partly completed matrix **X**.



$$\mathbf{X} = \begin{array}{c} \\ \text{From} \end{array}\begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

(i) Copy and complete the matrix **X**.

A second ferry company also offers ferry crossings between these four cities. The following matrix represents the total number of direct ferry crossings offered by the two ferry companies.

$$\begin{pmatrix} 0 & 2 & 3 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}$$

(ii) Find the matrix **Y** representing the ferry crossings offered by the second ferry company.

(iii) Draw a diagram similar to the one above, showing the ferry crossings offered by the second ferry company.

4 Find the values of w, x, y and z such that

$$\begin{pmatrix} 3 & w \\ -1 & 4 \end{pmatrix} + x\begin{pmatrix} 2 & -1 \\ y & z \end{pmatrix} = \begin{pmatrix} -9 & 8 \\ 11 & -8 \end{pmatrix}.$$

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

5 Find the possible values of p and q such that

$$\begin{pmatrix} p^2 & -3 \\ 2 & 9 \end{pmatrix} - \begin{pmatrix} 5p & -2 \\ -7 & q^2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 9 & 4 \end{pmatrix}.$$

M **6** Four local football teams took part in a competition in which every team plays each of the others twice, once at home and once away. The results matrix after half of the games had been played is:

	Win	Draw	Lose	Goals for	Goals against
Stars	2	1	0	6	3
Cougars	0	0	3	2	8
Town	2	0	1	4	3
United	1	1	1	5	3

(i) The results of the next three matches are as follows:

Stars	2	Cougars	0
Town	3	United	3
Stars	2	Town	4

Find the results matrix for these three matches and hence find the complete results matrix for all the matches so far.

(ii) Here is the complete results matrix for the whole competition.

$$\begin{pmatrix} 4 & 1 & 1 & 12 & 8 \\ 1 & 1 & 4 & 5 & 12 \\ 3 & 1 & 2 & 12 & 10 \\ 1 & 3 & 2 & 10 & 9 \end{pmatrix}$$

Find the results matrix for the last three matches (Stars vs United, Cougars vs Town and Cougars vs United) and deduce the result of each of these three matches.

M **7** A mail-order clothing company stocks a jacket in three different sizes and four different colours.

The matrix $\mathbf{P} = \begin{pmatrix} 17 & 8 & 10 & 15 \\ 6 & 12 & 19 & 3 \\ 24 & 10 & 11 & 6 \end{pmatrix}$ represents the number of jackets in stock at the start of one week.

The matrix $\mathbf{Q} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 1 & 3 & 4 & 6 \\ 5 & 0 & 2 & 3 \end{pmatrix}$ represents the number of orders for jackets received during the week.

(i) Find the matrix $\mathbf{P} - \mathbf{Q}$.

What does this matrix represent? What does the negative element in the matrix mean?

A delivery of jackets is received from the manufacturers during the week.

The matrix $\mathbf{R} = \begin{pmatrix} 5 & 10 & 10 & 5 \\ 10 & 10 & 5 & 15 \\ 0 & 0 & 5 & 5 \end{pmatrix}$ shows the number of jackets received.

(ii) Find the matrix that represents the number of jackets in stock at the end of the week after all the orders have been dispatched.

(iii) Assuming that this week is typical, find the matrix that represents sales of jackets over a six-week period. How realistic is this assumption?

1.2 Multiplication of matrices

When you multiply two matrices you do not just multiply corresponding terms. Instead you follow a slightly more complicated procedure. The following example will help you to understand the rationale for the way it is done.

There are four ways of scoring points in rugby: a try (five points), a conversion (two points), a penalty (three points) and a drop goal (three points). In a match, Tonga scored three tries, one conversion, two penalties and one drop goal.

So their score was

$3 \times 5 + 1 \times 2 + 2 \times 3 + 1 \times 3 = 26.$

You can write this information using matrices. The tries, conversions, penalties and drop goals that Tonga scored are written as the 1×4 row matrix $(3 \quad 1 \quad 2 \quad 1)$ and the points for the different methods of scoring as the 4×1 column matrix $\begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix}$.

These are combined to give the 1×1 matrix $(3 \times 5 + 1 \times 2 + 2 \times 3 + 1 \times 3) = (26)$.

Combining matrices in this way is called **matrix multiplication** and this example is written as $(3 \quad 1 \quad 2 \quad 1) \times \begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix} = (26)$.

The use of matrices can be extended to include the points scored by the other team, Japan. They scored two tries, two conversions, four penalties and one drop goal. This information can be written together with Tonga's scores as a 2×4 matrix, with one row for Tonga and the other for Japan. The multiplication is then written as

$$\begin{pmatrix} 3 & 1 & 2 & 1 \\ 2 & 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 26 \\ 29 \end{pmatrix}.$$

So Japan scored 29 points and won the match.

This example shows you two important points about matrix multiplication. Look at the orders of the matrices involved.

The two 'middle' numbers, in this case 4, must be the same for it to be possible to multiply two matrices. If two matrices can be multiplied, they are conformable for multiplication.

$$2 \times 4 \quad \times \quad 4 \times 1$$

The two 'outside' numbers give you the order of the product matrix, in this case 2×1.

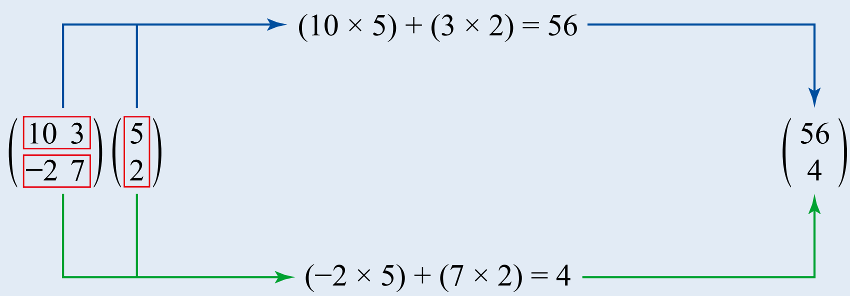
You can see from the previous example that multiplying matrices involves multiplying each element in a row of the left-hand matrix by each element in a column of the right-hand matrix and then adding these products.

Example 1.1

Find $\begin{pmatrix} 10 & 3 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Solution

The product will have order 2×1.



▲ **Figure 1.2**

Example 1.2

Find $\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 0 \\ -2 & -3 & 1 \end{pmatrix}$.

Solution

The order of this product is 2×3.

- $(1 \times 4) + (3 \times -2) = -2$
- $(1 \times 3) + (3 \times -3) = -6$
- $(1 \times 0) + (3 \times 1) = 3$
- $(-2 \times 4) + (5 \times -2) = -18$
- $(-2 \times 3) + (5 \times -3) = -21$
- $(-2 \times 0) + (5 \times 1) = 5$

$$\begin{pmatrix} -2 & -6 & 3 \\ -18 & -21 & 5 \end{pmatrix}$$

So $\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 0 \\ -2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -6 & 3 \\ -18 & -21 & 5 \end{pmatrix}$

?

› If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 4 & 1 \\ 0 & 3 & 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 8 & -1 \\ -2 & 3 \\ 4 & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 & 0 \\ 3 & -4 \end{pmatrix}$, which of the products **AB**, **BA**, **AC**, **CA**, **BC** and **CB** exist?

Example 1.3

Find $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

What do you notice?

Solution

The order of this product is 2×2.

- $(3 \times 1) + (2 \times 0) = 3$
- $(3 \times 0) + (2 \times 1) = 2$
- $(-1 \times 0) + (4 \times 1) = 4$
- $(-1 \times 1) + (4 \times 0) = -1$

$$\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

Multiplying a matrix by the identity matrix has no effect.

Properties of matrix multiplication

In this section you will look at whether matrix multiplication is:

- commutative
- associative.

On page 4 you saw that for numbers, addition is both associative and commutative. Multiplication is also both associative and commutative. For example:

$$(3 \times 4) \times 5 = 3 \times (4 \times 5)$$

and

$$3 \times 4 = 4 \times 3$$

ACTIVITY 1.1

Using $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}$ find the products **AB** and **BA** and hence comment on whether or not matrix multiplication is commutative. Find a different pair of matrices, **C** and **D**, such that **CD** = **DC**.

Technology note

You could use the matrix function on your calculator.

ACTIVITY 1.2

Using $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, find the matrix products:

(i) **AB**

(ii) **BC**

(iii) (**AB**)**C**

(iv) **A**(**BC**)

Does your answer suggest that matrix multiplication is associative? Is this true for all 2 × 2 matrices? How can you prove your answer?

Exercise 1B

In this exercise, do not use a calculator unless asked to. A calculator can be used for checking answers.

1 Write down the orders of these matrices.

(i) (a) $\mathbf{A} = \begin{pmatrix} 3 & 4 & -1 \\ 0 & 2 & 3 \\ 1 & 5 & 0 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 2 & 3 & 6 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} 4 & 9 & 2 \\ 1 & -3 & 0 \end{pmatrix}$ (d) $\mathbf{D} = \begin{pmatrix} 0 & 2 & 4 & 2 \\ 0 & -3 & -8 & 1 \end{pmatrix}$

(e) $\mathbf{E} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ (f) $\mathbf{F} = \begin{pmatrix} 2 & 5 & 0 & -4 & 1 \\ -3 & 9 & -3 & 2 & 2 \\ 1 & 0 & 0 & 10 & 4 \end{pmatrix}$

(ii) Which of the following matrix products can be found? For those that can, state the order of the matrix product.

(a) **AE** (b) **AF** (c) **FA** (d) **CA** (e) **DC**

2 Calculate these products.

(i) $\begin{pmatrix} 3 & 0 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 4 & -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & -3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 5 & 8 \\ -3 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 5 & -1 & 0 \\ 3 & 6 & 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -9 \\ 11 \\ -2 \end{pmatrix}$

Check your answers using the matrix function on a calculator if possible.

CP **3** Using the matrices $\mathbf{A} = \begin{pmatrix} 5 & 9 \\ -2 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 5 \\ 2 & -9 \end{pmatrix}$, confirm that matrix multiplication is not commutative.

4 For the matrices

$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} -3 & 7 \\ 2 & 5 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 1 \end{pmatrix}$

$\mathbf{D} = \begin{pmatrix} 3 & 4 \\ 7 & 0 \\ 1 & -2 \end{pmatrix}$ $\mathbf{E} = \begin{pmatrix} 4 & 7 \\ 3 & -2 \\ 1 & 5 \end{pmatrix}$ $\mathbf{F} = \begin{pmatrix} 3 & 7 & -5 \\ 2 & 6 & 0 \\ -1 & 4 & 8 \end{pmatrix}$

calculate, where possible, the following:

(i) **AB** (ii) **BA** (iii) **CD** (iv) **DC** (v) **EF** (vi) **FE**

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

5 Using the matrix function on a calculator, find $\mathbf{M}^4$ for the matrix

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 4 & 3 \end{pmatrix}.$$

> **Note**
>
> $\mathbf{M}^4$ means $\mathbf{M} \times \mathbf{M} \times \mathbf{M} \times \mathbf{M}$

PS

6 $\mathbf{A} = \begin{pmatrix} x & 3 \\ 0 & -1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 2x & 0 \\ 4 & -3 \end{pmatrix}$:

(i) Find the matrix product **AB** in terms of x.

(ii) If $\mathbf{AB} = \begin{pmatrix} 10x & -9 \\ -4 & 3 \end{pmatrix}$, find the possible values of x.

(iii) Find the possible matrix products **BA**.

7 (i) For the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, find

(a) $\mathbf{A}^2$

(b) $\mathbf{A}^3$

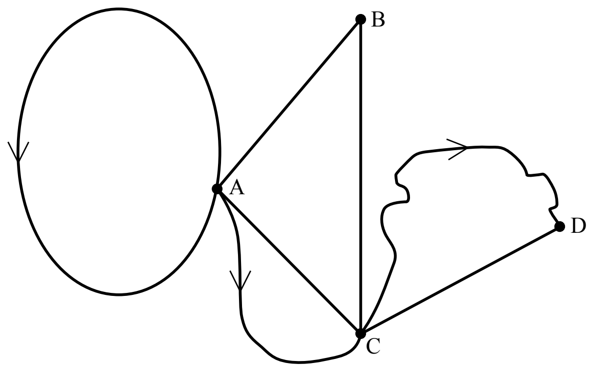
(c) $\mathbf{A}^4$

(ii) Suggest a general form for the matrix $\mathbf{A}^n$ in terms of n.

(iii) Verify your answer by finding $\mathbf{A}^{10}$ on your calculator and confirming it gives the same answer as (ii).

8 The map below shows the bus routes in a holiday area. Lines represent routes that run each way between the resorts. Arrows indicated one-way scenic routes.

M is the partly completed 4×4 matrix that shows the number of direct routes between the various resorts.



$$\mathbf{M} = \begin{array}{c} \\ \text{From} \end{array} \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array} \overset{\displaystyle\text{To}}{\begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \\ \end{array}} \begin{pmatrix} 1 & 1 & 2 & 0 \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

(i) Copy and complete the matrix **M**.

(ii) Calculate $\mathbf{M}^2$ and explain what information it contains.

(iii) What information would $\mathbf{M}^3$ contain?

9 $\mathbf{A} = \begin{pmatrix} 4 & x & 0 \\ 2 & -3 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 2 & -5 \\ 4 & x \\ x & 7 \end{pmatrix}$:

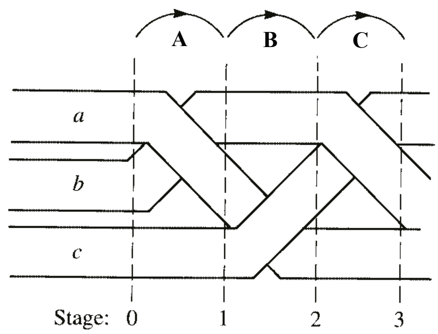
(i) Find the product **AB** in terms of x.

A symmetric matrix is one in which the entries are symmetrical about the leading diagonal, for example $\begin{pmatrix} 2 & 5 \\ 5 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 4 & -6 \\ 4 & 2 & 5 \\ -6 & 5 & 1 \end{pmatrix}$.

(ii) Given that the matrix **AB** is symmetric, find the possible values of x.

(iii) Write down the possible matrices **AB**.

PS **10** The diagram on the right shows the start of the plaiting process for producing a leather bracelet from three leather strands a, b and c.



The process has only two steps, repeated alternately:

Step 1: cross the top strand over the middle strand

Step 2: cross the middle strand under the bottom strand.

At the start of the plaiting process, Stage 0, the order of the strands is given by $\mathbf{S}_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

(i) Show that pre-multiplying $\mathbf{S_0}$ by the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ gives $\mathbf{S_1}$, the matrix that represents the order of the strands at Stage 1.

(ii) Find the 3×3 matrix **B** that represents the transition from Stage 1 to Stage 2.

(iii) Find matrix $\mathbf{M} = \mathbf{BA}$ and show that $\mathbf{MS_0}$ gives $\mathbf{S_2}$, the matrix that represents the order of the strands at Stage 2.

(iv) Find $\mathbf{M^2}$ and hence find the order of the strands at Stage 4.

(v) Calculate $\mathbf{M^3}$. What does this tell you?

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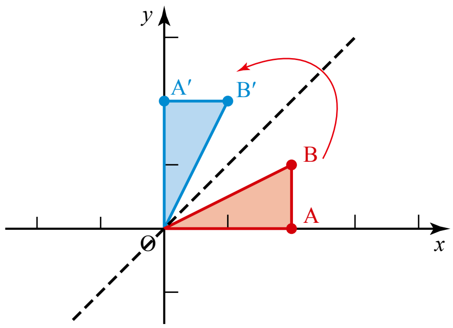
1.3 Transformations

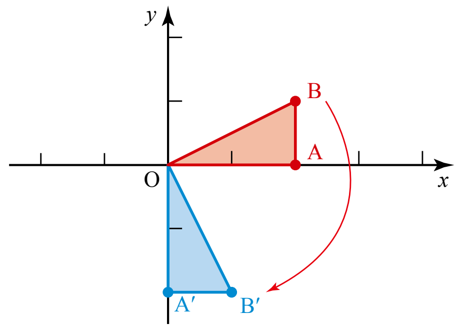
You are already familiar with several different types of transformation, including reflections, rotations and enlargements.

- The original point, or shape, is called the **object**.
- The new point, or shape, after the transformation, is called the **image**.
- A transformation is a **mapping** of an object onto its image.

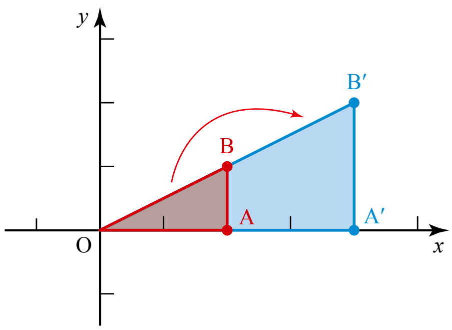
Some examples of transformations are illustrated in Figures 1.3 to 1.5 (note that the vertices of the image are denoted by the same letters with a dash, e.g. A′, B′).



▲ **Figure 1.3 Reflection in the line $y = x$**



▲ **Figure 1.4** Rotation through 90° clockwise, centre O



▲ **Figure 1.5** Enlargement centre O, scale factor 2

In this section, you will also meet the idea of

- a **stretch** parallel to the x-axis or y-axis
- a **shear**.

A transformation maps an object according to a rule and can be represented by a matrix (see next section). The effect of a transformation on an object can be found by looking at the effect it has on the **position vector** of the point $\begin{pmatrix} x \\ y \end{pmatrix}$, i.e. the vector from the origin to the point (x, y). So, for example, to find the effect of a transformation on the point (2, 3) you would look at the effect that the transformation matrix has on the position vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Vectors that have length or **magnitude** of 1 are called **unit vectors**.

In two dimensions, two unit vectors that are of particular interest are

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the x-axis

$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ – a unit vector in the direction of the y-axis.

The equivalent unit vectors in three dimensions are

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the x-axis

$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the y-axis

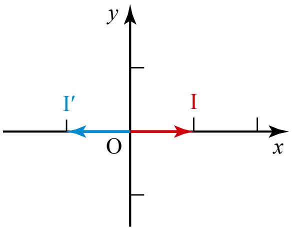
$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ – a unit vector in the direction of the z-axis.

Finding the transformation represented by a given matrix

Start by looking at the effect of multiplying the unit vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

The image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under this transformation is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$



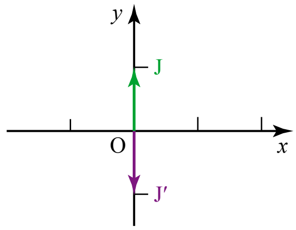
▲ **Figure 1.6**

Note

The letter I is often used for the point $(1, 0)$.

The image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under the transformation is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$



▲ **Figure 1.7**

> **Note**
>
> The letter J is often used for the point (0, 1).

You can see from this that the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a rotation, centre the origin, through 180°.

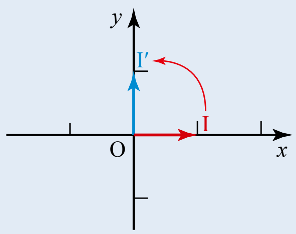
Example 1.4

Describe the transformations represented by the following matrices.

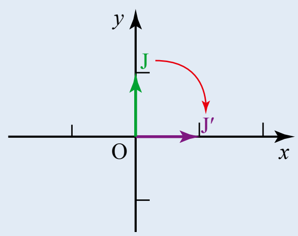
(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Solution

(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



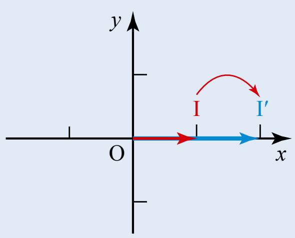
▲ **Figure 1.8**



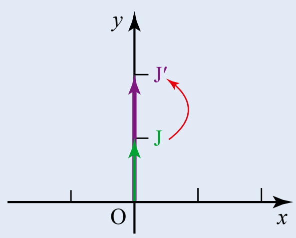
▲ **Figure 1.9**

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$.

(ii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$



▲ **Figure 1.10**



▲ **Figure 1.11**

The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement, centre the origin, scale factor 2.

You can see that the images of $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ are the two columns of the transformation matrix.

Finding the matrix that represents a given transformation

The connection between the images of the unit vectors **i** and **j** and the matrix representing the transformation provides a quick method for finding the matrix representing a transformation.

You may find it easier to see what the transformation is when you use a shape, like the unit square, rather than points or lines.

It is common to use the unit square with coordinates O (0, 0), I (1, 0), P (1, 1) and J (0, 1).

You can think about the images of the points I and J, and from this you can write down the images of the unit vectors **i** and **j**.

This is done in the next example.

Example 1.5

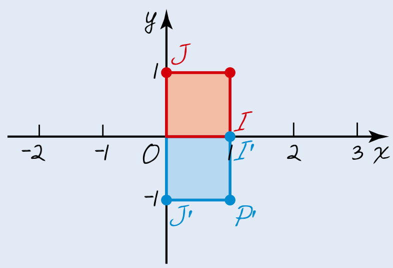
By drawing a diagram to show the image of the unit square, find the matrices that represent each of the following transformations:

(i) a reflection in the x-axis

(ii) an enlargement of scale factor 3, centre the origin.

Solution

(i)



▲ **Figure 1.12**

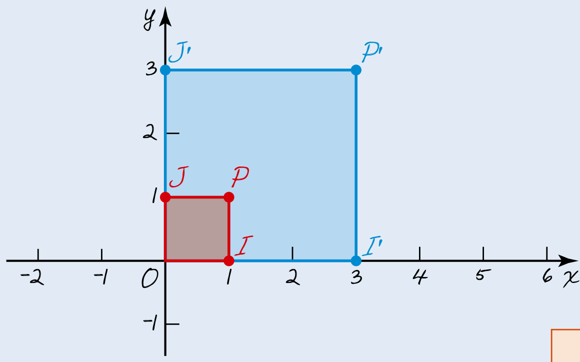
You can see from Figure 1.12 that I (1, 0) is mapped to itself and J (0, 1) is mapped to J′ (0, −1).

So the image of **I** is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and the image of **J** is $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

So the matrix that represents a reflection in the x-axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(ii)



▲ Figure 1.13

So the image of **I** is $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

You can see from Figure 1.13 that I (1, 0) is mapped to I′ (3, 0), and J (0, 1) is mapped to J′ (0, 3).

and the image of **J** is $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

So the matrix that represents an enlargement, centre the origin, scale factor 3 is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

?

› For a general transformation represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, what are the images of the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

› What is the image of the origin (0, 0)?

ACTIVITY 1.3

Using the image of the unit square, find the matrix which represents a rotation of 45° anticlockwise about the origin.
Use your answer to write down the matrices that represent the following transformations:

(i) a rotation of 45° clockwise about the origin

(ii) a rotation of 135° anticlockwise about the origin.

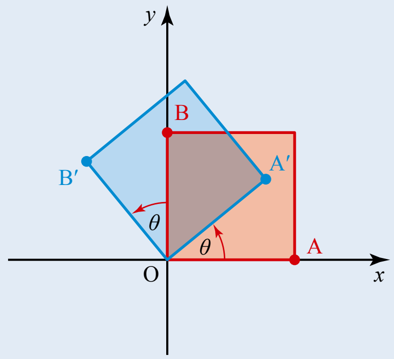
Example 1.6

(i) Find the matrix that represents a rotation through angle θ anticlockwise about the origin.

(ii) Use your answer to find the matrix that represents a rotation of 60° anticlockwise about the origin.

Solution

(i) Figure 1.14 shows a rotation of angle θ anticlockwise about the origin.



▲ **Figure 1.14**

Call the coordinates of the point A′ (p, q). Since the lines OA and OB are perpendicular, the coordinates of B′ will be $(-q, p)$.

From the right-angled triangle with OA′ as the hypotenuse, $\cos\theta = \frac{p}{1}$ and so $p = \cos\theta$.

Similarly, from the right-angled triangle with OB′ as the hypotenuse, $\sin\theta = \frac{q}{1}$ so $q = \sin\theta$.

So, the image point A′ (p, q) has position vector $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and the image point B′ $(-q, p)$ has position vector $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

Therefore, the matrix that represents a rotation of angle θ anticlockwise about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(ii) The matrix that represents an anticlockwise rotation of 60° about the origin is $\begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

? What matrix would represent a rotation through an angle θ clockwise about the origin?

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

ACTIVITY 1.4

Investigate the effect of the matrices:

(i) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

Describe the general transformation represented by the matrices $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$.

Technology note

You could use geometrical software to try different values of m and n.

Activity 1.4 illustrates two important general results:

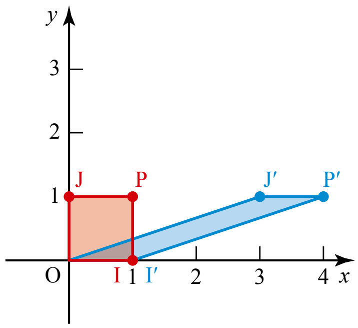
- The matrix $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ represents a stretch of scale factor m parallel to the x-axis.
- The matrix $\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$ represents a stretch of scale factor n parallel to the y-axis.

Shears

Figure 1.15 shows the unit square and its image under the transformation represented by the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ on the unit square. The matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ transforms the unit vector $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and transforms the unit vector $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

The point with position vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is transformed to the point with position vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

As $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$



▲ **Figure 1.15**

This transformation is called a **shear**. Notice that the points on the x-axis stay the same, and the points J and P move parallel to the x-axis to the right.

This shear can be described fully by saying that the x-axis is fixed, and giving the image of one point not on the x-axis, e.g. (0, 1) is mapped to (3, 1).

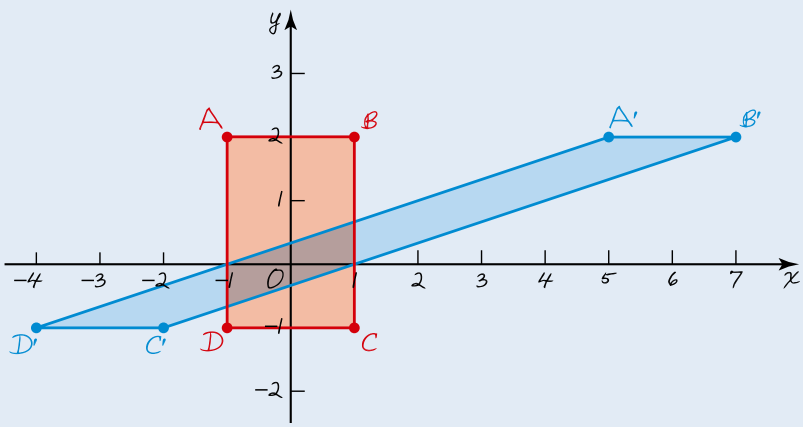
Generally, a shear with the x-axis fixed has the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and a shear with the y-axis fixed has the form $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

Example 1.7

Find the image of the rectangle with vertices A (–1, 2), B (1, 2), C (1, –1) and D (–1, –1) under the shear $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and show the rectangle and its image on a diagram.

Solution

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 & -1 \\ 2 & 2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 7 & -2 & -4 \\ 2 & 2 & -1 & -1 \end{pmatrix}$$



▲ **Figure 1.16**

→

The effect of this shear is to transform the sides of the rectangle parallel to the y-axis into sloping lines. Notice that the gradient of the side A′D′ is $\frac{1}{3}$, which is the reciprocal of the top right-hand element of the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

Note

Notice that under the shear transformation, points above the x-axis move to the right and points below the x-axis move to the left.

ACTIVITY 1.5

For each of the points A, B, C and D in Example 1.7, find

$$\frac{\text{distance between the point and its image}}{\text{distance of original point from } x\text{-axis}}.$$

What do you notice?

In the activity above, you should have found that dividing the distance between the point and its image by the distance of the original point from the x-axis (which is fixed), gives the answer 3 for all points, which is the number in the top right of the matrix. This is called the **shear factor** for the shear.

Technology note

If you have access to geometrical software, investigate how shears are defined.

!

There are different conventions about the sign of a shear factor, and for this reason shear factors are not used to define a shear in this book. It is possible to show the effect of matrix transformations using some geometrical computer software packages. You might find that some packages use different approaches towards shears and define them in different ways.

Example 1.8

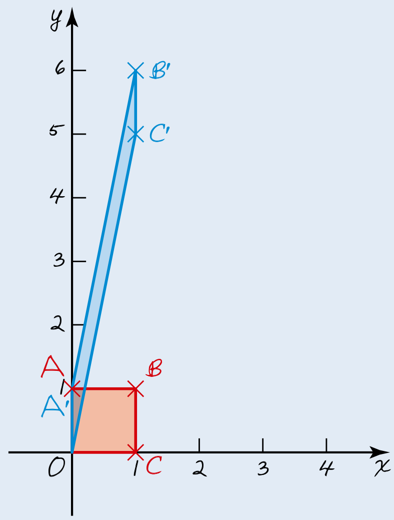
In a shear, S, the y-axis is fixed, and the image of the point (1, 0) is the point (1, 5).

(i) Draw a diagram showing the image of the unit square under the transformation S.

(ii) Find the matrix that represents the shear S.

Solution

(i)



▲ **Figure 1.17**

(ii) Under S $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ← Since the y-axis is fixed.

So the matrix representing S is $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$.

Notice that this matrix is of the form $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ for shears with the y-axis fixed.

Summary of transformations in two dimensions

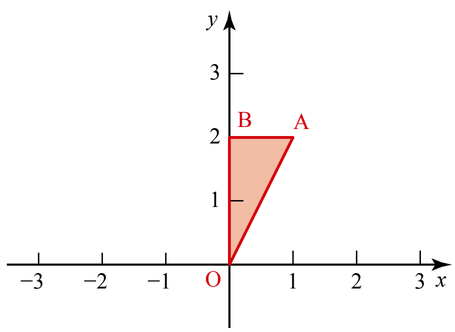
Reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Reflection in the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Rotation anticlockwise about the origin through angle θ	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	Enlargement, centre the origin, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Stretch parallel to the x-axis, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$	Stretch parallel to the y-axis, scale factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Shear, x-axis fixed, with (0, 1) mapped to (k, 1)	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	Shear, y-axis fixed, with (1, 0) mapped to (1, k)	$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

Note

All these transformations are examples of linear transformations. In a linear transformation, straight lines are mapped to straight lines, and the origin is mapped to itself.

Exercise 1C

1 The diagram shows a triangle with vertices at O, A (1, 2) and B (0, 2).



For each of the transformations below

(a) draw a diagram to show the effect of the transformation on triangle OAB

(b) give the coordinates of A′ and B′, the images of points A and B

(c) find expressions for x' and y', the coordinates of P′, the image of a general point P (x, y)

(d) find the matrix that represents the transformation.

(i) Enlargement, centre the origin, scale factor 3

(ii) Reflection in the x-axis

(iii) Reflection in the line $x + y = 0$

(iv) Rotation 90° clockwise about O

(v) Two-way stretch, scale factor 3 horizontally and scale factor $\frac{1}{2}$ vertically.

2 Describe the geometrical transformations represented by these matrices.

(i) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

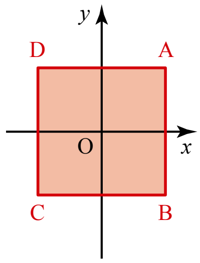
(iv) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ (v) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3 Each of the following matrices represents a rotation about the origin. Find the angle and direction of rotation in each case.

(i) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (ii) $\begin{pmatrix} 0.574 & -0.819 \\ 0.819 & 0.574 \end{pmatrix}$

(iii) $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ (iv) $\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

4 The diagram below shows a square with vertices at the points A (1, 1), B (1, –1), C (–1, –1) and D (–1,1).



(i) Draw a diagram to show the image of this square under the transformation matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(ii) Describe fully the transformation represented by the matrix **M**. State the fixed line and the image of the point A.

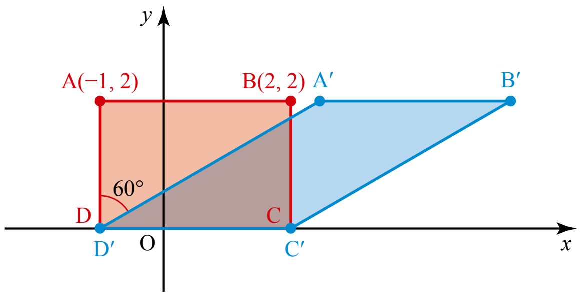
5 (i) Find the image of the unit square under the transformations represented by the matrices

(a) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$.

(ii) Use your answers to part (i) to fully describe the transformations represented by each of the matrices **A** and **B**.

6 The diagram below shows a shear that maps the rectangle ABCD to the parallelogram A′B′C′D′.

The angle A′DA is 60°.



(i) Find the coordinates of A′.

(ii) Find the matrix that represents the shear.

7 The unit square OABC has its vertices at (0, 0), (1, 0), (1, 1) and (0, 1).

OABC is mapped to OA′B′C′ by the transformation defined by the matrix $\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$.

Find the coordinates of A′, B′ and C′ and show that the area of the shape has not been changed by the transformation.

8 The transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is applied to the triangle ABC with vertices A (−1, 1), B (1, −1) and C (−1, −1).

(i) Draw a diagram showing the triangle ABC and its image A′B′C′.

(ii) Find the gradient of the line A′C′ and explain how this relates to the matrix **M**.

PS **9** A transformation maps P to P′ as follows:

- Each point is mapped on to the line $y = x$.
- The line joining a point to its image is parallel to the y-axis.

Find the coordinates of the image of the point (x, y) and hence show that this transformation can be represented by means of a matrix.

What is that matrix?

PS **10** A square has corners with coordinates A (1, 0), B (1, 1), C (0, 1) and O (0, 0). It is to be transformed into another quadrilateral in the first quadrant of the coordinate grid.

Find a matrix that would transform the square into:

(i) a rectangle with one vertex at the origin, the sides lie along the axes and one side of length is 5 units

(ii) a rhombus with one vertex at the origin, two angles of 45° and side lengths of $\sqrt{2}$ units; one of the sides lies along an axis

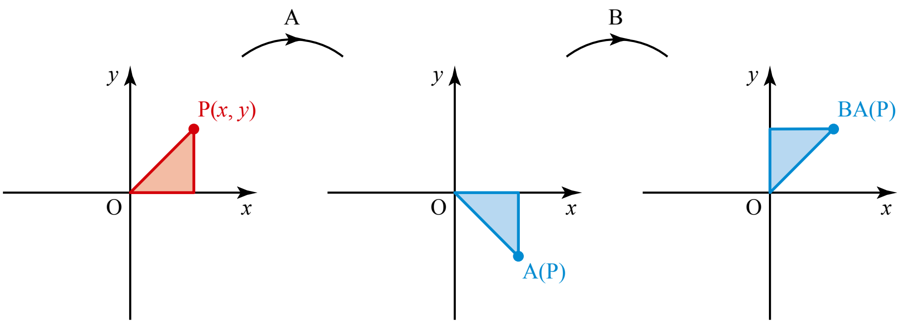
(iii) a parallelogram with one vertex at the origin and two angles of 30°; one of the longest sides lies along an axis and has length 7 units; the shortest sides have length 3 units.

Is there more than one possibility for any of these matrices? If so, write down alternative matrices that satisfy the same description.

1.4 Successive transformations

Figure 1.18 shows the effect of two successive transformations on a triangle. The transformation A represents a reflection in the x-axis. A maps the point P to the point A(P).

The transformation B represents a rotation of 90° anticlockwise about O. When you apply B to the image formed by A, the point A(P) is mapped to the point $B(A(\text{P}))$. This is abbreviated to BA(P).



▲ **Figure 1.18**

Note

Notice that a transformation written as BA means 'carry out A, then carry out B'.

This process is sometimes called **composition of transformations**.

?

Look at Figure 1.18 and compare the original triangle with the final image after both transformations.

- Describe the single transformation represented by BA.
- Write down the matrices which represent the transformations A and B. Calculate the matrix product **BA** and comment on your answer.

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

1

> **Note**
>
> A transformation is often denoted by a capital letter. The matrix representing this transformation is usually denoted by the same letter, in bold.

> **Technology note**
>
> If you have access to geometrical software, you could investigate this using several different matrices for **T** and **S**.

In general, the matrix for a composite transformation is found by multiplying the matrices of the individual transformations in reverse order. So, for two transformations the matrix representing the first transformation is on the right and the matrix for the second transformation is on the left. For n transformations $T_1, T_2, \ldots, T_{n-1}, T_n$, the matrix product would be $\mathbf{T}_n\mathbf{T}_{n-1} \ldots \mathbf{T}_2\mathbf{T}_1$.

You will prove this result for two transformations in Activity 1.6.

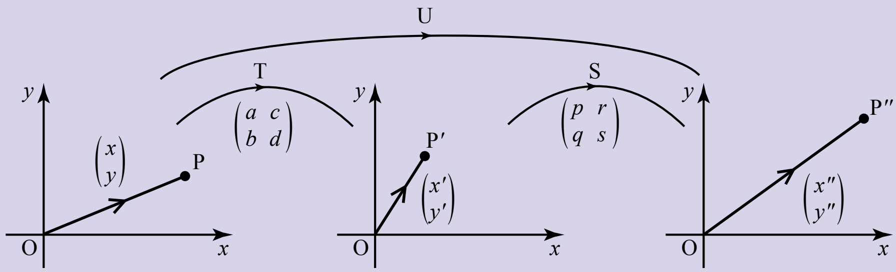
ACTIVITY 1.6

The transformations T and S are represented by the matrices

$\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

T is applied to the point P with position vector $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$. The image of P is P′.

S is then applied to the point P′. The image of P′ is P″. This is illustrated in Figure 1.19.



▲ **Figure 1.19**

(i) Find the position vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ of P′ by calculating the matrix product $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix}$.

(ii) Find the position vector $\begin{pmatrix} x'' \\ y'' \end{pmatrix}$ of P″ by calculating the matrix product $\mathbf{S}\begin{pmatrix} x' \\ y' \end{pmatrix}$.

(iii) Find the matrix product $\mathbf{U} = \mathbf{ST}$ and show that $\mathbf{U}\begin{pmatrix} x \\ y \end{pmatrix}$ is the same as $\begin{pmatrix} x'' \\ y'' \end{pmatrix}$.

?

› How can you use the idea of successive transformations to explain the associativity of matrix multiplication $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$?

Proving results in trigonometry

If you carry out a rotation about the origin through angle θ, followed by a rotation about the origin through angle ϕ, then this is equivalent to a single rotation about the origin through angle $\theta + \phi$. Using matrices to represent these transformations allows you to prove the formulae for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$. This is done in Activity 1.7.

Note

Assume that a rotation is anticlockwise unless otherwise stated

ACTIVITY 1.7

(i) Write down the matrix **A** representing a rotation about the origin through angle θ, and the matrix **B** representing a rotation about the origin through angle ϕ.

(ii) Find the matrix **BA**, representing a rotation about the origin through angle θ, followed by a rotation about the origin through angle ϕ.

(iii) Write down the matrix **C** representing a rotation about the origin through angle $\theta + \phi$.

(iv) By equating **C** to **BA**, write down expressions for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$.

(v) Explain why **BA** = **AB** in this case.

Example 1.9

(i) Write down the matrix **A** that represents an anticlockwise rotation of 135° about the origin.

(ii) Write down the matrices **B** and **C** that represent rotations of 45° and 90° respectively about the origin. Find the matrix **BC** and verify that **A** = **BC**.

(iii) Calculate the matrix $\mathbf{B}^3$ and comment on your answer.

Solution

(i) $$\mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(ii) $$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \mathbf{A}$$

→

Answers to exercises are available at www.hoddereducation.com/cambridgeextras

(iii) $\mathbf{B}^3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

This verifies that three successive anticlockwise rotations of 45° about the origin is equivalent to a single anticlockwise rotation of 135° about the origin.

Exercise 1D

1 $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(i) Describe the transformations that are represented by matrices **A**, **B**, **C** and **D**.

(ii) Find the following matrix products and describe the single transformation represented in each case:

(a) **BC** (b) **CB** (c) **DC** (d) $\mathbf{A}^2$ (e) **BCB** (f) $\mathbf{DC}^2\mathbf{D}$

(iii) Write down two other matrix products, using the matrices **A**, **B**, **C** and **D**, which would produce the same single transformation as $\mathbf{DC}^2\mathbf{D}$.

2 The matrix **X** represents a reflection in the x-axis.

The matrix **Y** represents a reflection in the y-axis.

(i) Write down the matrices **X** and **Y**.

(ii) Find the matrix **XY** and describe the transformation it represents.

(iii) Find the matrix **YX**.

(iv) Explain geometrically why **XY** = **YX** in this case.

PS **3** The matrix **P** represents a rotation of 180° about the origin.

The matrix **Q** represents a reflection in the line $y = x$.

(i) Write down the matrices **P** and **Q**.

(ii) Find the matrix **PQ** and describe the transformation it represents.

(iii) Find the matrix **QP**.

(iv) Explain geometrically why **PQ** = **QP** in this case.

4 The transformations R and S are represented by the matrices

$\mathbf{R} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix}$.

(i) Find the matrix which represents the transformation RS.

(ii) Find the image of the point (3, -2) under the transformation RS.

PS **5** The transformation represented by $\mathbf{C} = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$ is equivalent to a single transformation B followed by a single transformation A. Give geometrical descriptions of a pair of possible transformations B and A and state the matrices that represent them.

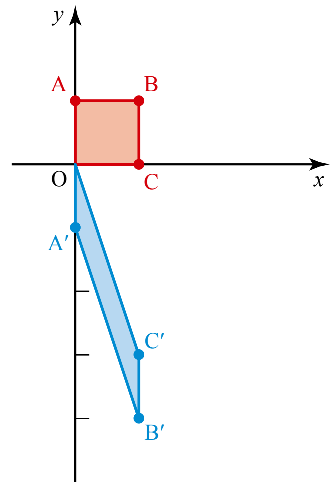
Comment on the order in which the transformations are performed.

6 The diagram on the right shows the image of the unit square OABC under the combined transformation with matrix **PQ**.

(i) Write down the matrix **PQ**.

Matrix **P** represents a reflection.

(ii) State the matrices **P** and **Q** and define fully the two transformations represented by these matrices. When describing matrix **Q** you should refer to the image of the point B.



PS **7** Find the matrix **X** that represents rotation of 135° about the origin followed by a reflection in the y-axis.

Explain why matrix **X** cannot represent a rotation about the origin.

Note

Assume that a rotation is anticlockwise unless otherwise stated

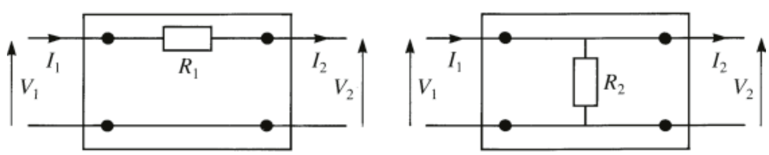
PS **8** (i) Write down the matrix **P** that represents a stretch of scale factor 2 parallel to the y-axis.

(ii) The matrix $\mathbf{Q} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$. Write down the two single transformations that are represented by the matrix **Q**.

(iii) Find the matrix **PQ**. Write a list of the three transformations that are represented by the matrix **PQ**. In how many different orders could the three transformations occur?

(iv) Find the matrix **R** for which the matrix product **RPQ** would transform an object to its original position.

PS **9** There are two basic types of four-terminal electrical networks, as shown in the diagrams below.



In Type A the output voltage V_2 and current I_2 are related to the input voltage V_1 and current I_1 by the simultaneous equations:

$$V_2 = V_1 - I_1R_1$$
$$I_2 = I_1$$

The simultaneous equations can be written as $\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \mathbf{A}\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$.

(i) Find the matrix **A**.

In Type B the corresponding simultaneous equations are:

$$V_2 = V_1$$

$$I_2 = I_1 - \frac{V_1}{R_2}$$

(ii) Write down the matrix **B** that represents the effect of a Type B network.

(iii) Find the matrix that represents the effect of Type A followed by Type B.

(iv) Is the effect of Type B followed by Type A the same as the effect of Type A followed by Type B?

10 The matrix **B** represents a rotation of 45° anticlockwise about the origin.

$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where a and b are positive real numbers

Given that $\mathbf{D}^2 = \mathbf{B}$, find exact values for a and b. Write down the transformation represented by the matrix **D**. What do the exact values a and b represent?

In questions **11** and **12** you will need to use the matrix that represents a reflection in the line $y = mx$. This can be written as $\frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$.

11 (i) Find the matrix **P** that represents reflection in the line $y = \frac{1}{\sqrt{3}}x$, and the matrix **Q** that represents reflection in the line $y = \sqrt{3}x$.

(ii) Use matrix multiplication to find the single transformation equivalent to reflection in the line $y = \frac{1}{\sqrt{3}}x$ followed by reflection in the line $y = \sqrt{3}x$.

Describe this transformation fully.

(iii) Use matrix multiplication to find the single transformation equivalent to reflection in the line $y = \sqrt{3}x$ followed by reflection in the line $y = \frac{1}{\sqrt{3}}x$.

Describe this transformation fully.

CP **12** The matrix **R** represents a reflection in the line $y = mx$.

Show that $\mathbf{R}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and explain geometrically why this is the case.

1.5 Invariance

Invariant points

- In a reflection, are there any points that map to themselves?
- In a rotation, are there any points that map to themselves?

Points that map to themselves under a transformation are called **invariant points**. The origin is always an invariant point under a transformation that can be represented by a matrix, as the following statement is always true:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

More generally, a point (x, y) is invariant if it satisfies the matrix equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

For example, the point $(-2, 2)$ is invariant under the transformation represented by the matrix $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}$: $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

Example 1.10

M is the matrix $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

(i) Show that $(5, 5)$ is an invariant point under the transformation represented by **M**.

(ii) What can you say about the invariant points under this transformation?

Solution

(i) $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ so $(5, 5)$ is an invariant point under the transformation represented by **M**.

(ii) Suppose the point $\begin{pmatrix} x \\ y \end{pmatrix}$ maps to itself. Then

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x - y \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Leftrightarrow 2x - y = x$ and $x = y$.

Both equations simplify to $y = x$.

So the invariant points of the transformation are all the points on the line $y = x$.

These points all have the form (λ, λ). The point $(5,5)$ is just one of the points on this line.

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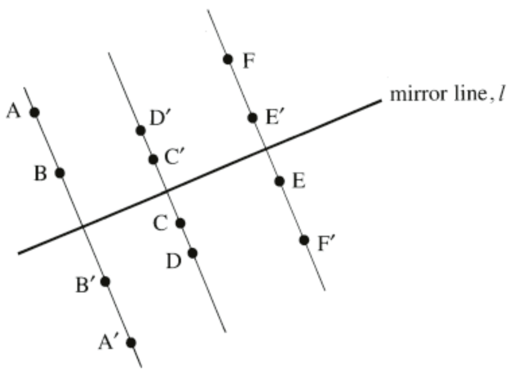
The simultaneous equations in Example 1.10 were equivalent and so all the invariant points were on a straight line. Generally, any matrix equation set up to find the invariant points will lead to two equations of the form $ax + by = 0$, which can also be expressed in the form $y = -\frac{ax}{b}$. These equations may be equivalent, in which case this is a line of invariant points. If the two equations are not equivalent, the origin is the only point that satisfies both equations, and so this is the only invariant point.

Invariant lines

A line AB is known as an **invariant line** under a transformation if the image of every point on AB is also on AB. It is important to note that it is not necessary for each of the points to map to itself; it can map to itself or to some other point on the line AB.

Sometimes it is easy to spot which lines are invariant. For example, in Figure 1.20 the position of the points A–F and their images A′–F′ show that the transformation is a reflection in the line l. So every point on l maps onto itself and l is a **line of invariant points**.

Look at the lines perpendicular to the mirror line in Figure 1.20, for example the line ABB′A′. Any point on one of these lines maps onto another point on the same line. Such a line is invariant but it is not a line of invariant points.



▲ **Figure 1.20**

Example 1.11

Find the invariant lines of the transformation given by the matrix $\mathbf{M} = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$.

Solution

Suppose the invariant line has the form $y = mx + c$

> Let the original point be (x, y) and the image point be (x', y').

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow x' = 5x + y \text{ and } y' = 2x + 4y$$

$$\Leftrightarrow \begin{cases} x' = 5x + mx + c = (5 + m)x + c \\ y' = 2x + 4(mx + c) = (2 + 4m)x + 4c \end{cases}$$

> Using $y = mx + c$.

As the line is invariant, (x', y') also lies on the line, so $y' = mx' + c$.

Therefore,

$(2 + 4m)x + 4c = m[(5 + m)x + c] + c$

$\Leftrightarrow 0 = (m^2 + m - 2)x + (m - 3)c$

For the left-hand side to equal zero, both $m^2 + m - 2 = 0$ and $(m - 3)c = 0$.

$(m - 1)(m + 2) = 0 \Leftrightarrow m = 1$ or $m = -2$

and

$(m - 3)c = 0 \Leftrightarrow m = 3$ or $c = 0$

$m = 3$ is not a viable solution as $m^2 + m - 2 \neq 0$.

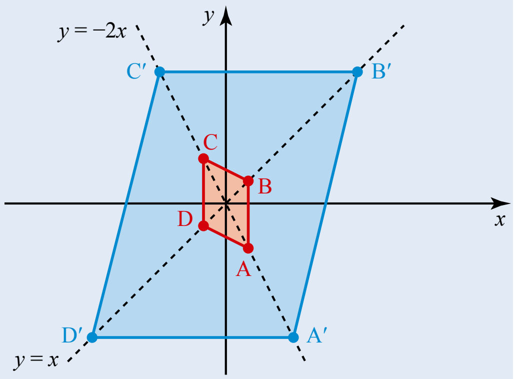
So, there are two possible solutions for the invariant line:

$m = 1, c = 0 \Leftrightarrow y = x$

or

$m = -2, c = 0 \Leftrightarrow y = -2x$

Figure 1.21 shows the effect of this transformation, together with its invariant lines.



▲ **Figure 1.21**

Exercise 1E

1 Find the invariant points under the transformations represented by the following matrices.

(i) $\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$ (iv) $\begin{pmatrix} 7 & -4 \\ 3 & -1 \end{pmatrix}$

2 What lines, if any, are invariant under the following transformations?

(i) Enlargement, centre the origin

(ii) Rotation through 180° about the origin

(iii) Rotation through 90° about the origin

(iv) Reflection in the line $y = x$

(v) Reflection in the line $y = -x$

(vi) Shear, x-axis fixed

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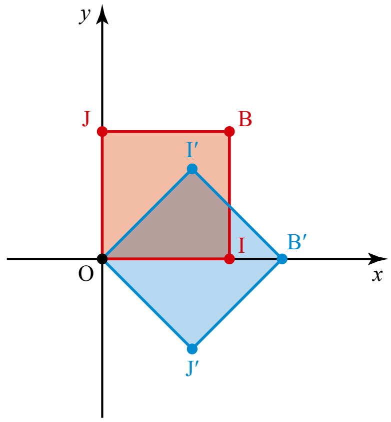
3 The diagram below shows the effect on the unit square of a transformation represented by $\mathbf{A} = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$.



(i) Find three points that are invariant under this transformation.

(ii) Given that this transformation is a reflection, write down the equation of the mirror line.

(iii) Using your answer to part (ii), write down the equation of an invariant line, other than the mirror line, under this reflection.

(iv) Justify your answer to part (iii) algebraically.

4 For the matrix $\mathbf{M} = \begin{pmatrix} 4 & 11 \\ 11 & 4 \end{pmatrix}$

(i) show that the origin is the only invariant point

(ii) find the invariant lines of the transformation represented by **M**.

5 (i) Find the invariant lines of the transformation given by the matrix $\begin{pmatrix} 3 & 4 \\ 9 & -2 \end{pmatrix}$.

(ii) Draw a diagram to show the effect of the transformation on the unit square, and show the invariant lines on your diagram.

6 For the matrix $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

(i) find the line of invariant points of the transformation given by **M**

(ii) find the invariant lines of the transformation

(iii) draw a diagram to show the effect of the transformation on the unit square.

CP **7** The matrix $\begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$ represents a reflection in the line $y = mx$.

Prove that the line $y = mx$ is a line of invariant points.

CP 8 The transformation T maps $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Show that invariant points other than the origin exist if $ad - bc = a + d - 1$.

PS 9 T is a translation of the plane by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. The point (x, y) is mapped to the point (x', y').

(i) Write down equations for x' and y' in terms of x and y.

(ii) Verify that $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ produces the same equations as those obtained in part (i).

The point (X, Y) is the image of the point (x, y) under the combined transformation TM where

$$\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 & a \\ 0.8 & 0.6 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

(iii) (a) Show that if $a = -4$ and $b = 2$ then $(0, 5)$ is an invariant point of TM.

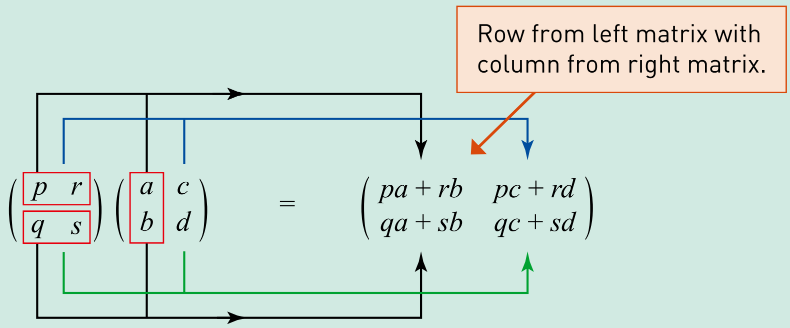
(b) Show that if $a = 2$ and $b = 1$ then TM has no invariant point.

(c) Find a relationship between a and b that must be satisfied if TM is to have any invariant points.

KEY POINTS

1. A matrix is a rectangular array of numbers or letters.
2. The shape of a matrix is described by its order. A matrix with r rows and c columns has order $r \times c$.
3. A matrix with the same number of rows and columns is called a square matrix.
4. The matrix $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is known as the 2×2 zero matrix. Zero matrices can be of any order.
5. A matrix of the form $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is known as an identity matrix. All identity matrices are square, with 1s on the leading diagonal and zeros elsewhere.
6. Matrices can be added or subtracted if they have the same order.
7. Two matrices **A** and **B** can be multiplied to give matrix **AB** if their orders are of the form $p \times q$ and $q \times r$ respectively. The resulting matrix will have the order $p \times r$.

8 Matrix multiplication



9 Matrix addition and multiplication are associative:

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

10 Matrix addition is commutative but matrix multiplication is generally not commutative:

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

$\mathbf{AB} \neq \mathbf{BA}$

11 The matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the transformation that maps the point with position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to the point with position vector $\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$.

12 A list of the matrices representing common transformations, including rotations, reflections, enlargements, stretches and shears, is given on page 24.

13 Under the transformation represented by **M**, the image of $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the first column of **M** and the image of $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the second column of **M**.

14 The composite of the transformation represented by M followed by that represented by N is represented by the matrix product **NM**.

15 If (x, y) is an invariant point under a transformation represented by the matrix **M**, then $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

16 A line AB is known as an invariant line under a transformation if the image of every point on AB is also on AB.

Note

Work on matrices is developed further in Chapter 6 'Matrices and their inverses'.

LEARNING OUTCOMES

Now that you have finished this chapter, you should be able to

- understand what is meant by the terms
  - order of a matrix
  - square matrix
  - zero matrix
  - equal matrices
- carry out the matrix operations
  - addition
  - subtraction
  - multiplication by a scalar
- understand when matrices are conformable for multiplication and be able to carry out matrix multiplication
- use a calculator to carry out matrix operations
- understand the use of matrices to represent the geometric transformations in the x–y plane
  - rotation about the origin
  - reflection in lines through the origin
  - enlargement with centre the origin
  - stretch parallel to the coordinate axes
  - shear with the axes as fixed lines
- recognise that the matrix product **AB** represents the transformation that results from the transformation represented by **B** followed by the transformation represented by **A**
- find the matrix that represents a given transformation or sequence of transformations
- understand the meaning of 'invariant' in the context of transformations represented by matrices
  - as applied to points
  - as applied to lines
- solve simple problems involving invariant points and invariant lines, for example
  - locate the invariant points of the transformation
  - find the invariant lines of the transformation
  - show lines of a given gradient are invariant for a certain transformation.