

1 The curve C has equation $y = x^a$ for $0 \leqslant x \leqslant 1$, where a is a positive constant. Find, in terms of a, the coordinates of the centroid of the region enclosed by C, the line $x = 1$ and the x-axis. [6]

2 It is given that $y = \ln(ax + 1)$, where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{d^n y}{dx^n} = (-1)^{n-1}\frac{(n-1)!a^n}{(ax+1)^n}.$$

[6]

3 The integral I_n, where n is a positive integer, is defined by

$$I_n = \int_{\frac{1}{2}}^{1} x^{-n} \sin \pi x \, dx.$$

(i) Show that

$$n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n. \qquad [5]$$

(ii) Find I_5 in terms of π and I_1. [2]

4 The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

....

....

....

....

....

....

....

(ii) State the equation of the other asymptote of C. [1]

....

....

(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

5 Let $S_N = \sum_{r=1}^{N}(5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N}\frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2 + 90N + 83).$$ [3]

(ii) Use the method of differences to express T_N in terms of N. [4]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

(iii) Find $\lim_{N\to\infty}\left(N^{-3}S_N T_N\right)$. [2]

....

....

....

....

....

....

....

....

....

....

6 With O as the origin, the points A, B, C have position vectors

$$\mathbf{i} - \mathbf{j}, \quad 2\mathbf{i} + \mathbf{j} + 7\mathbf{k}, \quad \mathbf{i} - \mathbf{j} + \mathbf{k}$$

respectively.

(i) Find the shortest distance between the lines OC and AB. [5]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of the lines OC and AB. [4]

7 The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. [4]

(ii) Show that $\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2\alpha^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \frac{58}{49}$. [3]

....

....

....

....

....

....

....

....

....

....

....

....

(iii) Find the exact value of $\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\gamma^3\alpha^3} + \frac{\gamma^3}{\alpha^3\beta^3}$. [2]

....

....

....

....

....

....

....

....

....

....

8 The matrix **M** is defined by

$$\mathbf{M} = \begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix},$$

where $m \neq 0, 1, 2$.

(i) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$. [7]

(ii) Find $\mathbf{M}^7\mathbf{P}$. [3]

9 **(i)** Use de Moivre's theorem to show that

$$\sec 6\theta = \frac{\sec^6\theta}{32 - 48\sec^2\theta + 18\sec^4\theta - \sec^6\theta}.$$ [6]

(ii) Hence obtain the roots of the equation

$$3x^6 - 36x^4 + 96x^2 - 64 = 0$$

in the form $\sec q\pi$, where q is rational. [5]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

10 The matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 \\ 1 & -2 & -2 \\ 2 & 3 & \theta \end{pmatrix}.$$

(i) (a) Find the rank of $\mathbf{A}$ when $\theta \neq -1$. [3]

....

....

....

....

....

....

....

....

....

....

....

(b) Find the rank of $\mathbf{A}$ when $\theta = -1$. [1]

....

....

....

Consider the system of equations

$$\begin{aligned} x + 5y + z &= -1, \\ x - 2y - 2z &= 0, \\ 2x + 3y + \theta z &= \theta. \end{aligned}$$

(ii) Solve the system of equations when $\theta \neq -1$. [3]

....

....

....

(iii) Find the general solution when $\theta = -1$. [3]

(iv) Show that if $\theta = -1$ and $\phi \neq -1$ then $\mathbf{Ax} = \begin{pmatrix} -1 \\ 0 \\ \phi \end{pmatrix}$ has no solution. [2]

11 Answer only **one** of the following two alternatives.

EITHER

It is given that $w = \cos y$ and

$$\tan y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2\tan y \frac{dy}{dx} = 1 + e^{-2x} \sec y.$$

(i) Show that

$$\frac{d^2w}{dx^2} + 2\frac{dw}{dx} + w = -e^{-2x}.$$ [4]

....

....

....

....

....

....

....

....

....

(ii) Find the particular solution for y in terms of x, given that when $x = 0$, $y = \frac{1}{3}\pi$ and $\frac{dy}{dx} = \frac{1}{\sqrt{3}}$. [10]

....

....

....

....

....

....

....

....

OR

The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, as follows:

$$C_1 : r = 2(e^{\theta} + e^{-\theta}),$$
$$C_2 : r = e^{2\theta} - e^{-2\theta}.$$

The curves intersect at the point P where $\theta = \alpha$.

(i) Show that $e^{2\alpha} - 2e^{\alpha} - 1 = 0$. Hence find the exact value of α and show that the value of r at P is $4\sqrt{2}$. [6]

(ii) Sketch C_1 and C_2 on the same diagram. [3]

(iii) Find the area of the region enclosed by C_1, C_2 and the initial line, giving your answer correct to 3 significant figures. [5]

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

