 

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | $A=\int_0^1 x^a\,\mathrm{d}x=\left[\frac{x^{a+1}}{a+1}\right]_0^1=\frac{1}{a+1}$ | B1 | Finds area of region. |
| | $A\bar{x}=\int_0^1 xy\,\mathrm{d}x=\int_0^1 x^{a+1}\,\mathrm{d}x=\left[\frac{x^{a+2}}{a+2}\right]_0^1=\frac{1}{a+2}$ | M1 A1 | Finds $\int_0^1 xy\,\mathrm{d}x$ . |
| | $2A\bar{y}=\int_0^1 y^2\,\mathrm{d}x=\int_0^1 x^{2a}\,\mathrm{d}x=\left[\frac{x^{2a+1}}{2a+1}\right]_0^1=\frac{1}{2a+1}$ | M1 A1 | Finds $\int_0^1 y^2\,\mathrm{d}x$ . |
| | $(\bar{x},\bar{y})=\left(\frac{a+1}{a+2},\frac{a+1}{2(2a+1)}\right)$ | A1 | Both coordinates correct. |
| | | 6 | |



 

 

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| 2 | $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a}{ax+1}=(-1)^0\frac{0!a^1}{(ax+1)^1}$ so true for $n=1$. | **M1 A1** | Proves base case. |
| | Assume that $\frac{\mathrm{d}^k y}{\mathrm{d}x^k}=(-1)^{k-1}\frac{(k-1)!a^k}{(ax+1)^k}$ for some positive integer k. | **B1** | States inductive hypothesis. |
| | Then $\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}=-ka(-1)^{k-1}\frac{(k-1)!a^k}{(ax+1)^{k+1}}=(-1)^k\frac{k!a^{k+1}}{(ax+1)^{k+1}}$ so true for $n=k+1$. | **M1 A1** | Differentiates $k^{\text{th}}$ derivative. |
| | By induction, true for every positive integer $n$. | **A1** | States conclusion. |
| | | **6** | |



 

 

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| 3(i) | $I_{n+2} = \left[\frac{x^{-n-1}}{-n-1}\sin \pi x\right]_{\frac{1}{2}}^{1} - \pi\int_{\frac{1}{2}}^{1}\frac{x^{-n-1}}{-n-1}\cos \pi x\,\mathrm{d}x$ | **M1 A1** | Integrates by parts. |
| | $= \frac{2^{n+1}}{n+1} + \frac{\pi}{n+1}\left(\left[\frac{x^{-n}}{-n}\cos \pi x\right]_{\frac{1}{2}}^{1} + \pi\int_{\frac{1}{2}}^{1}\frac{x^{-n}}{-n}\sin \pi x\,\mathrm{d}x\right)$ | **M1** | Integrates by parts again. |
| | $= \frac{2^{n+1}}{n+1} + \frac{\pi}{n+1}\left(\frac{1}{n} - \frac{\pi}{n}I_n\right)$ <br> $\Rightarrow (n+1)I_{n+2} = 2^{n+1} + \pi\left(\frac{1}{n} - \frac{\pi}{n}I_n\right)$ | **M1** | Uses $I_n$. |
| | $\Rightarrow n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n$ | **A1** | AG |
| | | **5** | |
| 3(ii) | $2I_3 = 4 + \pi - \pi^2 I_1$ <br> $12I_5 = 48 + \pi - \frac{\pi^2}{2}\left(4 + \pi - \pi^2 I_1\right)$ | **M1** | Substitutes $I_3$ into reduction formula. |
| | $\Rightarrow I_5 = 4 + \frac{1}{24}\left(2\pi - 4\pi^2 - \pi^3 + \pi^4 I_1\right)$ | **A1** | AEF, must be exact with fractions simplified. |
| | | **2** | |



 

 

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| 4(i) | $x^2+1=(ax+b)(2x+1)+c$ | **M1** | Uses that $2x+1$ is the quotient. |
| | $\Rightarrow a=\frac{1}{2},\ b=-\frac{1}{4}$ | **A1 A1** | |
| | | **3** | |
| 4(ii) | $x=\frac{1}{2}$ | **B1 FT** | |
| | | **1** | |



 

 

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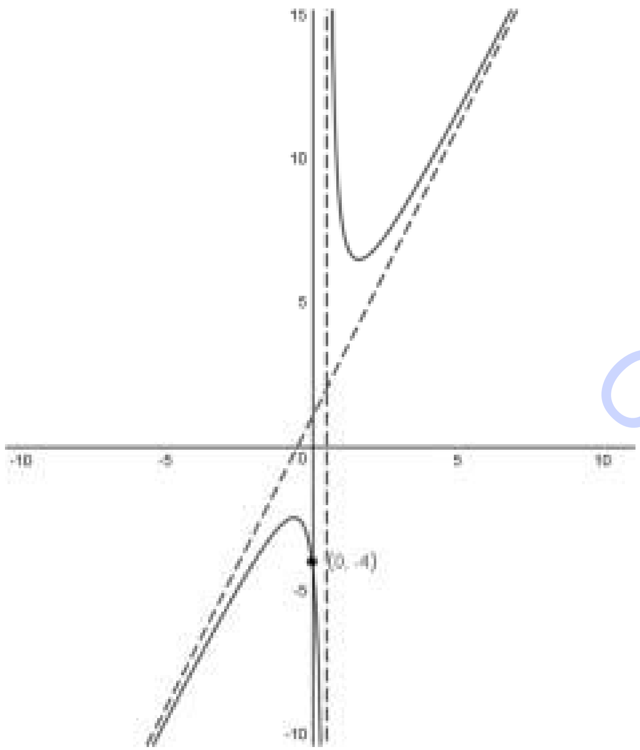
|---|---|---|---|
| 4(iii) |  | **B1** | Intersection (0,-4) given and asymptotes drawn. |
| | | **B1** | Left branch correct. |
| | | **B1 FT** | Right branch correct.<br><br>Deduct at most one mark for poor forms at infinity. |
| | | **3** | |



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| 5(i) | $\sum_{r=1}^{N}(5r+1)(5r+6)=25\sum_{r=1}^{N}r^2+35\sum_{r=1}^{N}r+6N$ | M1 | Expands. |
| | $25\left(\frac{1}{6}N(N+1)(2N+1)\right)+35\left(\frac{1}{2}N(N+1)\right)+6N$ | M1 | Substitutes formulae for $\sum r$ and $\sum r^2$. |
| | $=N\left(\frac{25}{6}(2N^2+3N+1)+\frac{35}{2}N+\frac{35}{2}+6\right)=\frac{1}{3}N(25N^2+90N+83)$ | A1 | Simplifies to the given answer (AG). |
| | | **3** | |
| 5(ii) | $\frac{1}{(5r+1)(5r+6)}=\frac{1}{5}\left(\frac{1}{5r+1}-\frac{1}{5r+6}\right)$ | M1 A1 | Finds partial fractions. |
| | $T_N=\frac{1}{5}\left(\frac{1}{6}-\frac{1}{11}+\frac{1}{11}-\frac{1}{16}+\cdots+\frac{1}{5N+1}-\frac{1}{5N+6}\right)$ | M1 | Expresses terms as differences. |
| | $\frac{1}{5}\left(\frac{1}{6}-\frac{1}{5N+6}\right)=\frac{1}{30}-\frac{1}{5(5N+6)}$ | A1 | At least 3 terms including last. |
| | | **4** | |
| 5(iii) | $\frac{S_N}{N^3}T_N\to\frac{25}{3}\times\frac{1}{30}=\frac{5}{18}$ | M1 A1 | Divides $S_N$ by $N^3$ and takes limits as $N\to\infty$ |
| | | **2** | |

 

 

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| 6(i) | $\overrightarrow{AB}=\begin{pmatrix}1\\2\\7\end{pmatrix}$ | **B1** | |
| | $\overrightarrow{OC}\times\overrightarrow{AB}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\1&2&7\end{vmatrix}=\begin{pmatrix}-9\\-6\\3\end{pmatrix}=t\begin{pmatrix}3\\2\\-1\end{pmatrix}$ | **M1 A1** | Finds direction of common perpendicular. |
| | $\dfrac{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-1\end{pmatrix}}{\sqrt{3^2+2^2+1^2}}=\dfrac{1}{\sqrt{14}}=0.267$ | **M1 A1** | Uses formula for shortest distance. |
| | | **5** | |
| 6(ii) | $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\3&2&-1\end{vmatrix}=t\begin{pmatrix}-1\\4\\5\end{pmatrix}$ | **M1 A1** | Finds normal to plane. |
| | $-(0)+4(0)+5(0)=0$ | **M1** | Uses point on plane. |
| | $-x+4y+5z=0$ | **A1** | AEF |
| | | **4** | |

 

 

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| 7(i) | $\sqrt{-7y}\left(-7y\right)+2\left(-7y\right)+\sqrt{-7y}+7=0$ <br> $\Rightarrow\sqrt{-7y}\left(-7y+1\right)=14y-7\Rightarrow -7y\left(-7y+1\right)^2=\left(14y-7\right)^2$ | M1 | Uses given substitution and eliminates radical. |
| | $\Rightarrow 49y^3+14y^2-27y+7=0$ | A1 | AG |
| | $y=\dfrac{x^2}{-7}=\dfrac{x^2}{\alpha\beta\gamma}$ | M1 | Uses $\alpha\beta\gamma=-7$. |
| | So roots are $\dfrac{\alpha^2}{\alpha\beta\gamma}=\dfrac{\alpha}{\beta\gamma},\ \dfrac{\beta^2}{\alpha\beta\gamma}=\dfrac{\beta}{\alpha\gamma},\ \dfrac{\gamma^2}{\alpha\beta\gamma}=\dfrac{\gamma}{\alpha\beta}$ | A1 | AG |
| | | 4 | |
| 7(ii) | $\dfrac{\alpha}{\beta\gamma}+\dfrac{\beta}{\alpha\gamma}+\dfrac{\gamma}{\alpha\beta}=-\dfrac{2}{7},\ \dfrac{1}{\gamma^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\alpha^2}=-\dfrac{27}{49}$ | B1 | States sum of roots and $\alpha'\beta'+\alpha'\gamma'+\beta'\gamma'$. |
| | $\dfrac{\alpha^2}{\beta^2\gamma^2}+\dfrac{\beta^2}{\alpha^2\gamma^2}+\dfrac{\gamma^2}{\alpha^2\beta^2}=\left(-\dfrac{2}{7}\right)^2-2\left(-\dfrac{27}{49}\right)=\dfrac{58}{49}$ | M1 A1 | Uses <br> $\alpha'^2+\beta'^2+\gamma'^2=\left(\alpha'+\beta'+\gamma'\right)^2-2\left(\alpha'\beta'+\alpha'\gamma'+\beta'\gamma'\right)$ <br> AG |
| | | 3 | |
| 7(iii) | $49\left(\dfrac{\alpha^3}{\beta^3\gamma^3}+\dfrac{\beta^3}{\alpha^3\gamma^3}+\dfrac{\gamma^3}{\alpha^3\beta^3}\right)=-14\left(\dfrac{58}{49}\right)+27\left(-\dfrac{2}{7}\right)-21$ | M1 | Uses $49\alpha'^3=-14\alpha'^2+27\alpha'-7$. |
| | $\Rightarrow\dfrac{\alpha^3}{\beta^3\gamma^3}+\dfrac{\beta^3}{\alpha^3\gamma^3}+\dfrac{\gamma^3}{\alpha^3\beta^3}=-\dfrac{317}{343}$ | A1 | |
| | | 2 | |

 

 

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| 8(i) | Eigenvalues of (upper diagonal matrix) $\mathbf{A}$ are $2, m$ and $1$.<br>(Or from characteristic equation: $(\lambda-2)(\lambda-m)(\lambda-1)=0$ ) | B1 | |
| | $\lambda=2:\ \mathbf{e}_1=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & m-2 & 1\\ 0 & 0 & -1\end{vmatrix}=\begin{pmatrix}2-m\\0\\0\end{pmatrix}=t\begin{pmatrix}1\\0\\0\end{pmatrix}$ | M1 A1 | Uses vector product (or equations) to find corresponding eigenvectors. |
| | $\lambda=m:\ \mathbf{e}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2-m & m & 1\\ 0 & 0 & 7\end{vmatrix}=\begin{pmatrix}7m\\7(m-2)\\0\end{pmatrix}=t\begin{pmatrix}m\\m-2\\0\end{pmatrix}$ | A1 | |
| | $\lambda=1:\ \mathbf{e}_3=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & m & 1\\ 0 & m-1 & 7\end{vmatrix}=\begin{pmatrix}6m+1\\-7\\m-1\end{pmatrix}=t\begin{pmatrix}6m+1\\-7\\m-1\end{pmatrix}$ | A1 | |
| | Thus $\mathbf{P}=\begin{pmatrix}1 & m & 6m+1\\ 0 & m-2 & -7\\ 0 & 0 & m-1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}2 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & 1\end{pmatrix}$ | M1 A1 FT | Or correctly matched permutations of columns. No follow through on two or more zero eigenvectors. |
| | | 7 | |



 

 

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| 8(ii) | $\mathbf{M}^7\mathbf{P} = \mathbf{P}\mathbf{D}^7\mathbf{P}^{-1}\mathbf{P} = \mathbf{P}\mathbf{D}^7 = \begin{pmatrix} 1 & m & 6m+1 \\ 0 & m-2 & -7 \\ 0 & 0 & m-1 \end{pmatrix}\begin{pmatrix} 2^7 & 0 & 0 \\ 0 & m^7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | **M1 A1 FT** | Applies $\mathbf{M}^7 = \mathbf{P}\mathbf{D}^7\mathbf{P}^{-1}$. |
| | $= \begin{pmatrix} 2^7 & m^8 & 6m+1 \\ 0 & m^8-2m^7 & -7 \\ 0 & 0 & m-1 \end{pmatrix}$ | **A1** | Order of columns might be swapped depending on **P**. |
| | | **3** | |

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| 9(i) | Write $c = \cos\theta$, $s = \sin\theta$.<br>$\cos 6\theta + i\sin 6\theta = (c+is)^6$ | **M1** | Uses binomial theorem. |
| | $\Rightarrow \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$ | **A1** | |
| | $c^6 - 15c^4s^2 + 15c^2s^4 - s^6 = c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$ | **M1** | Uses $c^2 = 1 - s^2$. |
| | $= c^6 - 15c^4(1-c^2) + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6)$ | **A1** | |
| | $= 32c^6 - 48c^4 + 18c^2 - 1$ | **M1** | Divides numerator and denominator by $c^6$. |
| | $\Rightarrow \sec 6\theta = \dfrac{1}{32c^6 - 48c^4 + 18c^2 - 1} = \dfrac{\sec^6\theta}{32 - 48\sec^2\theta + 18\sec^4\theta - \sec^6\theta}$ | **A1** | AG |
| | | **6** | |



 

 

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| 9(ii) | $x^6 = 2\left(32-48x^2+18x^4-x^6\right) \Rightarrow \dfrac{x^6}{32-48x^2+18x^4-x^6} = 2$ | **M1 A1** | Relates with equation in part (i). |
| | $\sec 6\theta = 2 \Rightarrow \cos 6\theta = \dfrac{1}{2}$ | **M1** | Solves $\cos 6\theta = \dfrac{1}{2}$. |
| | $x = \sec\dfrac{\pi}{18}$ | **A1** | Gives one correct solution. |
| | $x = \sec q\pi$ , $q = \dfrac{5}{18}, \dfrac{7}{18}, \dfrac{11}{18}, \dfrac{13}{18}, \dfrac{17}{18}$ | **A1** | Gives five other solutions. Allow different values of $q$ as long as all six solutions are found. |
| | | **5** | |



 

 

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| 10(i) | $\begin{pmatrix}1 & 5 & 1\\ 1 & -2 & -2\\ 2 & 3 & \theta\end{pmatrix} \to \begin{pmatrix}1 & 5 & 1\\ 0 & -7 & -3\\ 0 & 0 & \theta+1\end{pmatrix}$ | **M1 A1** | Reduces to echelon form. At least one row operation for M1. |
| | $r(\mathbf{A})=3$ if $\theta \neq -1$ | **A1** | |
| | $r(\mathbf{A})=2$ if $\theta = -1$ | **B1** | |
| | | **4** | |
| 10(ii) | $\begin{aligned} x \quad +5y \quad +z &= -1 \\ -7y \quad -3z &= 1 \\ (\theta+1)z &= (\theta+1) \end{aligned}$ | **M1** | Uses reduced form of augmented matrix or eliminates variables from scratch. |
| | $z=1,\ y=-\frac{4}{7},\ x=\frac{6}{7}$ | **A1**<br>**A1** | One correct.<br>All three correct. |
| | | **3** | |
| 10(iii) | $\begin{aligned} x \quad +5y \quad +z &= -1 \\ -7y \quad -3z &= 1 \\ (\theta+1)z &= (\theta+1) \end{aligned}$<br><br>$z=t$ | **M1** | Uses parameter. |
| | $y=-\frac{3t+1}{7},\ x=\frac{8t-2}{7}$ | **A1 A1** | |
| | | **3** | |

 

 

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| 10(iv) | $x+5y+z=-1$,<br>$-7y-3z=1$,<br>$(\theta+1)z=\phi+1$ | M1 | Uses reduced form of augmented matrix or eliminates variables from scratch |
| | $\theta=-1\Rightarrow\phi=-1$ so no solution (inconsistent). | A1 | |
| | | 2 | |



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| 11E(i) | $w=\cos y\Rightarrow\frac{\mathrm{d}w}{\mathrm{d}x}=-\sin y\frac{\mathrm{d}y}{\mathrm{d}x}$ | B1 | |
| | $\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=-\sin y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\cos y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ | B1 | |
| | $\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+2\frac{\mathrm{d}w}{\mathrm{d}x}+w=-\sin y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\cos y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-2\sin y\frac{\mathrm{d}y}{\mathrm{d}x}+\cos y$ | M1 | Uses substitution to obtain $w$ - $x$ equation, AG. |
| | $=-\cos y\left(\mathrm{e}^{-2x}\sec y\right)=-\mathrm{e}^{-2x}$ | A1 | |
| | | 4 | |

 

 

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| 11E(ii) | $m^2+2m+1=0 \Rightarrow m=-1$ | **M1** | Finds CF. |
| | CF: $w=(Ax+B)\mathrm{e}^{-x}$ | **A1** | |
| | PI: $w=k\mathrm{e}^{-2x} \Rightarrow w'=-2k\mathrm{e}^{-2x} \Rightarrow w''=4k\mathrm{e}^{-2x}$ | **M1** | Forms PI and differentiates. |
| | $4k-4k+k=-1 \Rightarrow k=-1$ | **A1** | |
| | $w=(Ax+B)\mathrm{e}^{-x}-\mathrm{e}^{-2x}$ | **A1** | States general solution. |
| | $x=0$, $y=\frac{1}{3}\pi$, $w=\frac{1}{2} \Rightarrow B=\frac{3}{2}$ | **B1** | Uses initial conditions to find constants. |
| | $w'=-(Ax+B)\mathrm{e}^{-x}+A\mathrm{e}^{-x}+2\mathrm{e}^{-2x}$ | **M1** | Differentiates general solution. |
| | $x=0$, $y=\frac{1}{3}\pi$, $y'=\frac{\sqrt{3}}{3}$, $w'=-\frac{1}{2} \Rightarrow -\frac{1}{2}=-\frac{3}{2}+A+2 \Rightarrow A=-1$ | **M1 A1** | Substitutes initial conditions. |
| | $y=\cos^{-1}\left(\left(\frac{3}{2}-x\right)\mathrm{e}^{-x}-\mathrm{e}^{-2x}\right)$ | **A1** | States particular solution for $y$ in terms of $x$. |
| | | **10** | |



 

 

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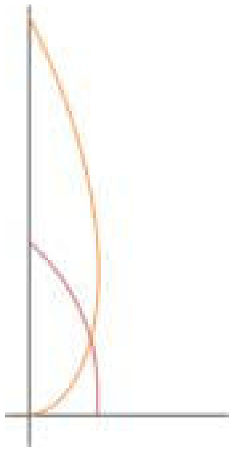
| 11O(i) | $e^{2\alpha} - e^{-2\alpha} = 2\left(e^{\alpha} + e^{-\alpha}\right) \Rightarrow e^{\alpha} - e^{-\alpha} = 2$ | M1 | Sets equations equal and divides by $e^{\alpha} + e^{-\alpha}$. |
| | $e^{2\alpha} - 2e^{\alpha} - 1 = 0 \Rightarrow e^{\alpha} = 1 + \sqrt{2}$ | M1 A1 | Forms quadratic in $e^{\alpha}$, AG. |
| | $\alpha = \ln\left(1+\sqrt{2}\right)$ | A1 | Must be exact. |
| | $r = 2\left(1+\sqrt{2}+\sqrt{2}-1\right) = 4\sqrt{2}$ | M1 A1 | Substitutes to find $r$. |
| | | **6** | |
| 11O(ii) |  | B1 | $C_1$ has correct shape. |
| | | B1 | $C_2$ has correct shape. |
| | | B1 | Intersection points positioned correctly. |
| | | **3** | |



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| 11O(iii) | $2\int_0^{\ln(1+\sqrt{2})}\left(e^{\theta}+e^{-\theta}\right)^2\ d\theta-\frac{1}{2}\int_0^{\ln(1+\sqrt{2})}\left(e^{2\theta}-e^{-2\theta}\right)^2\ d\theta$ <br> $=\int_0^{\ln(1+\sqrt{2})}5+2e^{2\theta}+2e^{-2\theta}-\frac{1}{2}e^{4\theta}-\frac{1}{2}e^{-4\theta}\ d\theta$ | M1 A1 | Uses $\frac{1}{2}\int r^2\ d\theta$ to formulate correct area. |
| | $=\left[5\theta+e^{2\theta}-e^{-2\theta}-\frac{1}{8}e^{4\theta}+\frac{1}{8}e^{-4\theta}\right]_0^{\ln(1+\sqrt{2})}$ | M1 A1 | Expands and integrates. |
| | $=5\ln(1+\sqrt{2})+\left(1+\sqrt{2}\right)^2-\left(1+\sqrt{2}\right)^{-2}-\frac{1}{8}\left(\left(1+\sqrt{2}\right)^4-\left(1+\sqrt{2}\right)^{-4}\right)=5.82$ | A1 | |
| | | 5 | |



 