

1 Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n. [5]

2 The curve C has polar equation $r^2 = \ln(1 + \theta)$, for $0 \leqslant \theta \leqslant 2\pi$.

(i) Sketch C. [2]

(ii) Using the substitution $u = 1 + \theta$, or otherwise, find the area of the region bounded by C and the initial line, leaving your answer in an exact form. [5]

3 **(i)** Write down the fifth roots of unity. [2]

(ii) Find all the roots of the equation

$$z^{10} + z^5 + 1 = 0,$$

giving each root in the form $e^{i\theta}$. [5]

4 **(i)** Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

(ii) Find the limit, as $N \to \infty$, of $\displaystyle\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

5 The linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is represented by the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 5 & 2 & 1 & -3 \\ 4 & 0 & 1 & -7 \\ -2 & 4 & -1 & \alpha \end{pmatrix}.$$

It is given that the rank of **M** is 2.

(i) Find the value of α and state a basis for the range space of T. [4]

(ii) Obtain a basis for the null space of T. [4]

6 The curve C has equation

$$y = \frac{x^2}{kx - 1},$$

where k is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the coordinates of the stationary points of C. [3]

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(iii) Sketch C. [3]

7 The line l_1 passes through the points $A(-3, 1, 4)$ and $B(-1, 5, 9)$. The line l_2 passes through the points $C(-2, 6, 5)$ and $D(-1, 7, 5)$.

(i) Find the shortest distance between the lines l_1 and l_2. [5]

(ii) Find the acute angle between the line l_2 and the plane containing A, B and D. [5]

8 Find the particular solution of the differential equation

$$9\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + x = 50\sin t,$$

given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$. [10]

9 A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$
$$\alpha\beta\gamma = -12,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

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(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

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(iii) Show that $b^3 - 15b + 126 = 0$. [4]

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(iv) Given that $3 + i\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

10 Let $I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^n x \, dx$, where $n \geqslant 0$.

(i) By considering $\frac{d}{dx}(\cot^{n+1} x)$, or otherwise, show that

$$I_{n+2} = \frac{1}{n+1} - I_n.$$ [5]

The curve C has equation $y = \cot x$, for $\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(ii) Find, in an exact form, the y-coordinate of the centroid of the region enclosed by C, the line $x = \frac{1}{4}\pi$ and the x-axis. [6]

11 Answer only **one** of the following two alternatives.

EITHER

A 3×3 matrix $\mathbf{A}$ has distinct eigenvalues 2, 1, 3, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

respectively, where b is a positive constant.

(i) Find $\mathbf{A}$ in terms of b. [9]

(ii) Find $\mathbf{A}^{-1}\begin{pmatrix}0\\2\\-2\end{pmatrix}$. [2]

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(iii) It is given that

$$\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}4\\4\\0\end{pmatrix} \quad \text{and} \quad \mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b^{-1}\end{pmatrix}.$$

Find the values of n and b. [3]

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OR

The positive variables y and t are related by

$$y = a^t,$$

where a is a positive constant.

(i) (a) By differentiating $\ln y$ with respect to t, show that $\dfrac{dy}{dt} = a^t \ln a$. [3]

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(b) Write down $\dfrac{d^2y}{dt^2}$. [1]

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(ii) Determine the set of values of a for which the infinite series

$$y + \frac{dy}{dt} + \frac{d^2y}{dt^2} + \frac{d^3y}{dt^3} + \ldots$$

is convergent. [3]

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A curve has parametric equations

$$x = t^a, \quad y = a^t.$$

(iii) Find $\frac{d^2y}{dx^2}$ in terms of a and t, and show that, when $t = 2$,

$$\frac{d^2y}{dx^2} = 2^{1-2a}(1 - a + 2\ln a)\ln a.$$ [7]

