

1 A curve C has equation $\cos y = x$, for $-\pi < x < \pi$.

(i) Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -\cot y\left(\frac{dy}{dx}\right)^2.$$ [4]

....

....

....

....

....

....

....

....

....

....

....

....

....

(ii) Hence find the exact value of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\pi\right)$ on C. [2]

....

....

....

....

....

....

....

....

2 Let $u_n = \dfrac{4\sin\left(n - \frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1) + \cos 1}$.

(i) Using the formulae for $\cos P \pm \cos Q$ given in the List of Formulae MF10, show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)}.$$ [2]

....

....

....

....

....

....

(ii) Use the method of differences to find $\sum_{n=1}^{N} u_n$. [2]

....

....

....

....

....

....

....

(iii) Explain why the infinite series $u_1 + u_2 + u_3 + \dots$ does not converge. [1]

....

....

....

....

....

3 The lines l_1 and l_2 have equations $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + \mu(-6\mathbf{j} + \mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

4 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^1 x^n e^{x^3} \, dx.$$

(i) Show that $I_2 = \frac{1}{3}(e - 1)$. [2]

....

....

....

....

....

....

....

(ii) Show that, for $n \geqslant 3$,

$$3I_n = e - (n - 2)I_{n-3}.$$ [3]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

(iii) Hence find the exact value of I_8. [3]

5 A curve C is defined parametrically by

$$x = \frac{2}{e^t + e^{-t}} \quad \text{and} \quad y = \frac{e^t - e^{-t}}{e^t + e^{-t}},$$

for $0 \leqslant t \leqslant 1$. The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by S.

(i) Show that $S = 4\pi \int_0^1 \frac{e^t - e^{-t}}{(e^t + e^{-t})^2}\, dt$. [5]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

(ii) Using the substitution $u = e^t + e^{-t}$, or otherwise, find S in terms of π and e. [3]

6 The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots α^3, β^3, γ^3. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

....

....

....

....

....

....

(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

7 Find the particular solution of the differential equation

$$10\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - x = t + 2,$$

given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$. [10]

8 **(i)** Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$ [5]

(ii) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty} \left(\tfrac{1}{2}\right)^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta}.$$ [5]

9 It is given that **e** is an eigenvector of the matrix **A**, with corresponding eigenvalue λ.

(i) Show that **e** is an eigenvector of $\mathbf{A}^2$, with corresponding eigenvalue λ^2. [2]

....

....

....

....

....

....

....

....

....

The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{A} + n\mathbf{I})^2,$$

where **I** is the 3×3 identity matrix and n is a non-zero integer.

(ii) Find, in terms of n, a non-singular matrix **P** and a diagonal matrix **D** such that $\mathbf{B} = \mathbf{PDP}^{-1}$. [8]

....

....

....

....

....

....

....

....

....

....

10 The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

(i) Find the equations of the asymptotes of C_1. [2]

....

....

....

....

....

(ii) Find the equation of the oblique asymptote of C_2. [2]

....

....

....

....

....

....

(iii) Show that C_1 and C_2 do not intersect. [2]

....

....

....

....

....

....

....

....

(iv) Find the coordinates of the stationary points of C_2. [3]

....

....

....

....

....

....

....

....

....

....

....

(v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]

11 Answer only **one** of the following two alternatives.

EITHER

The curve C_1 has polar equation $r^2 = 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta \tan \theta = 1$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r^2 = \theta \sec^2 \theta$, for $0 \leqslant \theta < \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(ii) Find the exact value of θ at Q. [2]

....

....

....

....

....

....

....

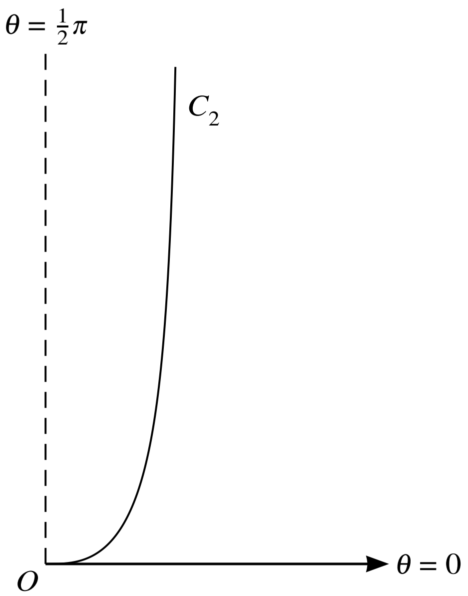
....

....

....

....

(iii) The diagram below shows the curve C_2. Sketch C_1 on this diagram. [2]



(iv) Find, in exact form, the area of the region OPQ enclosed by C_1 and C_2. [5]

OR

The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} -1 & 2 & 3 & 4 \\ 1 & 0 & 1 & -1 \\ 1 & -2 & -3 & a \\ 1 & 2 & 5 & 2 \end{pmatrix}.$$

(i) For $a \neq -4$, the range space of T is denoted by V.

(a) Find the dimension of V and show that

$$\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ -2 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ -1 \\ a \\ 2 \end{pmatrix}$$

form a basis for V. [5]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

(b) Show that if $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ belongs to V then $x + 2y = t$. [4]

(ii) For $a = -4$, find the general solution of

$$\mathbf{Mx} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

[5]

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

