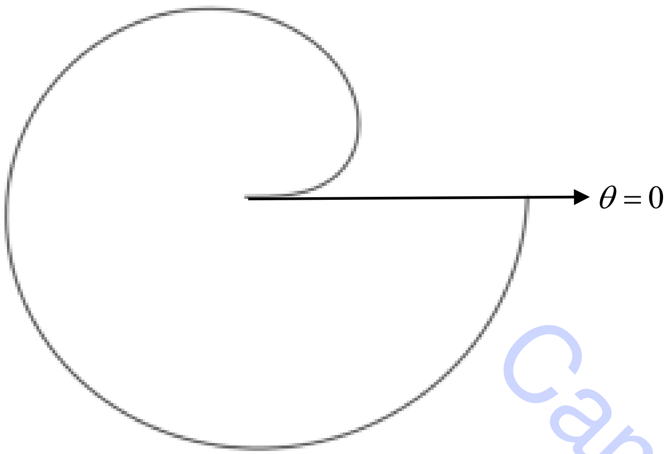


Question	Answer	Marks	Guidance
1	$3^3 - 1 = 26$ is divisible by 13	**B1**	Checks base case.
	Assume that $3^{3k} - 1$ is divisible by 13 for some positive integer k	**B1**	States inductive hypothesis.
	Then $3^{3k+3} - 1 = 3^3 3^{3k} - 1 = 26 \cdot 3^{3k} + 3^{3k} - 1$	**M1**	Separates $3^{3k} - 1$
	is divisible by 13	**A1**	
	$H_k \Rightarrow H_{k+1}$ By induction, $3^{3n} - 1$ is divisible by 13 for every positive integer n.	**A1**	States conclusion.
		5	

Question	Answer	Marks	Guidance
2(i)		B1	Correct shape, domain and orientation.
		B1	The initial line is tangential to C at the pole.
		2	
2(ii)	$A=\frac{1}{2}\int_0^{2\pi}\ln(1+\theta)\,\mathrm{d}\theta$	M1	States $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct expression and limits.
	$\int_0^{2\pi}\ln(1+\theta)\,\mathrm{d}\theta=\int_1^{2\pi+1}\ln u\ \mathrm{d}u$	M1	Applies given substitution correctly, changes their limits.
	$\int_1^{2\pi+1}\ln u\ \mathrm{d}u=[u\ln u]_1^{2\pi+1}-[u]_1^{2\pi+1}$	M1 A1	Integrates $\ln u$ (by parts or otherwise).
	$A=\left(\pi+\frac{1}{2}\right)\ln(2\pi+1)-\pi$	A1	AEF, must be exact.
		5	

Question	Answer	Marks	Guidance
3(i)	$\exp\left(i\frac{2\pi k}{5}\right),\ k=0,\pm1,\pm2$	**B2**	B1 for 1 correct fifth root. B1 for all 5 distinct, correct roots (AEF). SCB1 if only arguments are given.
3(ii)	$z^5=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$	**M1**	Correctly solves quadratic in z^5
	$z^5=\exp\left(i2\pi\left(\frac{1}{3}+k\right)\right)$ or $\exp\left(i2\pi\left(-\frac{1}{3}+k\right)\right)$	**M1 A1**	Writes in polar or exponential form and adds multiples of 2π
	$\exp\left(\pm i\frac{2\pi}{15}\right), \exp\left(\pm i\frac{4\pi}{15}\right), \exp\left(\pm i\frac{8\pi}{15}\right), \exp\left(\pm i\frac{2\pi}{3}\right), \exp\left(\pm i\frac{14\pi}{15}\right)$	**A2**	A1 for 5 distinct, correct roots. A1 for exactly 10 distinct, correct roots. Allow alternative exact values of θ such as $\theta=\frac{16\pi}{15},\frac{4\pi}{3},\frac{22\pi}{15},\frac{26\pi}{15},\frac{28\pi}{15}$.
	Alternative method for 3(ii)		
	$z^5+z^{-5}=-1$	**M1**	Divides through by z^5
	$2\cos 5\theta=-1$	**M1 A1**	Applies de Moivre's theorem
	$\exp\left(\pm i\frac{2\pi}{15}\right), \exp\left(\pm i\frac{4\pi}{15}\right), \exp\left(\pm i\frac{8\pi}{15}\right), \exp\left(\pm i\frac{2\pi}{3}\right), \exp\left(\pm i\frac{14\pi}{15}\right)$	**A2**	A1 for 5 distinct, correct roots. A1 for exactly 10 distinct, correct roots. Allow alternative exact values of θ such as $\theta=\frac{16\pi}{15},\frac{4\pi}{3},\frac{22\pi}{15},\frac{26\pi}{15},\frac{28\pi}{15}$.
		5	

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4(i)	$\dfrac{1}{(3r+1)(3r-2)}=\dfrac{1}{3}\left(\dfrac{1}{3r-2}-\dfrac{1}{3r+1}\right)$	M1 A1	Finds partial fractions.
	$\displaystyle\sum_{r=1}^{N}\frac{1}{(3r+1)(3r-2)}=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\ldots\frac{1}{3N-2}-\frac{1}{3N+1}\right)$	M1	At least 3 term including final term.
	$=\dfrac{1}{3}\left(1-\dfrac{1}{3N+1}\right)$	A1	AG
		4	
4(ii)	$\displaystyle\sum_{r=N+1}^{N^2}\frac{N}{(3r+1)(3r-2)}=\sum_{r=1}^{N^2}\frac{N}{(3r+1)(3r-2)}-\sum_{r=1}^{N}\frac{N}{(3r+1)(3r-2)}$	M1	Uses $\displaystyle\sum_{r=N+1}^{N^2}=\sum_{r=1}^{N^2}-\sum_{r=1}^{N}$
	$=\dfrac{N}{3}-\dfrac{N}{3(3N^2+1)}-\left(\dfrac{N}{3}-\dfrac{N}{3(3N+1)}\right)$	M1	Applies **(i)**
	$=\dfrac{N}{3(3N+1)}-\dfrac{N}{3(3N^2+1)}=\dfrac{N^3-N^2}{(3N+1)(3N^2+1)}$	A1	Allow simplification to common denominator.
	$\to\dfrac{1}{9}$ as $N\to\infty$	B1	
		4	

Question	Answer	Marks	Guidance
5(i)	$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 5 & 2 & 1 & -3 \\ 4 & 0 & 1 & -7 \\ -2 & 4 & -1 & \alpha \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -8 & 1 & -23 \\ 0 & -8 & 1 & -23 \\ 0 & 8 & -1 & \alpha+8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -8 & 1 & -23 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha-15 \end{pmatrix}$	**M1 A1**	Performs row operations correctly.
	$r(\mathbf{M}) = 2 \Rightarrow \alpha = 15$	**A1**	CWO
	Basis for the range space is $\left\{ \begin{pmatrix} 1 \\ 5 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix} \right\}$	**B1**	Accept any two independent column vectors from **M**.
		4	
5(ii)	$x + 2y \quad + 4t = 0$ $-8y + z - 23t = 0$	**M1**	Forms system of equations.
	$t = \lambda,\ z = \mu,\ y = \frac{1}{8}\mu - \frac{23}{8}\lambda,\ x = -\frac{1}{4}\mu + \frac{7}{4}\lambda$	**M1**	Uses two parameters.
	Basis for null space is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 14 \\ -23 \\ 0 \\ 8 \end{pmatrix} \right\}$	**A1 A1**	AEF. Many alternatives are possible e.g. $\left\{ \begin{pmatrix} -2 \\ 1 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 23 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 4 \\ 0 \\ -23 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -7 \\ -1 \end{pmatrix} \right\}$
		4	

Question	Answer	Marks	Guidance
6(i)	$x = \frac{1}{k}$	**B1**	States vertical asymptote.
	$y = \frac{(kx-1)(k^{-1}x + k^{-2}) + k^{-2}}{kx-1}$	**M1**	Finds oblique asymptote.
	Oblique asymptote is $y = k^{-1}x + k^{-2}$	**A1**	
		3	
6(ii)	$y' = \frac{2x(kx-1) - kx^2}{(kx-1)^2} = 0 \Rightarrow kx^2 - 2x = 0$	**M1**	Differentiates and equates to 0.
	$x = 0,\ 2k^{-1}$	**A1**	Finds x-coordinates.
	$(0,0),\ (2k^{-1}, 4k^{-2})$	**A1**	Finds y-coordinates
		3	

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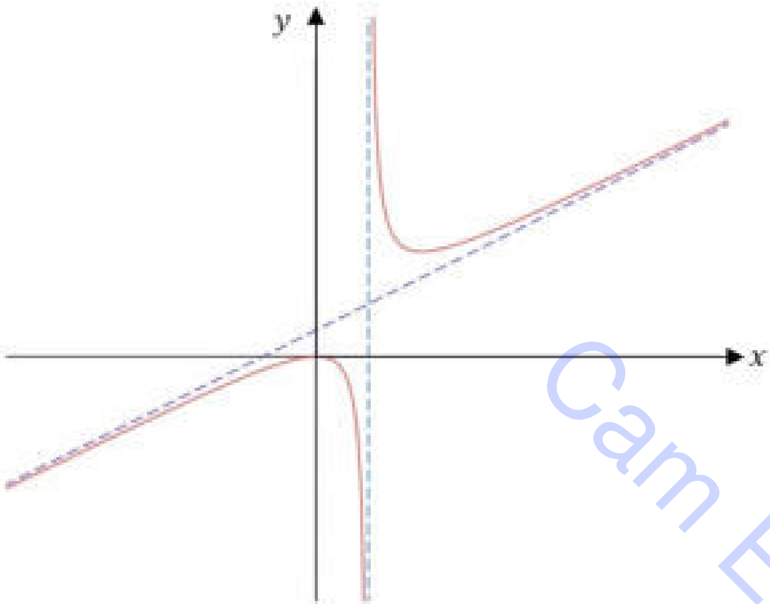
6(iii)		**B1**	Axes and asymptotes correct.
		B1	Upper branch correct.
		B1	Lower branch correct. Deduct at most 1 mark for poor forms at infinity.
		3	

Question	Answer	Marks	Guidance
7(i)	$\overrightarrow{AB}=\begin{pmatrix}2\\4\\5\end{pmatrix}$, $\overrightarrow{CD}=\begin{pmatrix}1\\1\\0\end{pmatrix}$	**B1**	Find the directions of the lines.
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 4 & 5\\1 & 1 & 0\end{vmatrix}=\begin{pmatrix}-5\\5\\-2\end{pmatrix}$	**M1 A1**	Finds direction of common perpendicular. Allow any non-zero scalar multiple.
	$\dfrac{\begin{pmatrix}1\\5\\1\end{pmatrix}\cdot\begin{pmatrix}-5\\5\\-2\end{pmatrix}}{\sqrt{5^2+5^2+2^2}}=\dfrac{18}{\sqrt{54}}=\sqrt{6}=2.45$	**M1 A1**	Uses formula for shortest distance.
		5	
7(ii)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 4 & 5\\2 & 6 & 1\end{vmatrix}=\begin{pmatrix}-26\\8\\4\end{pmatrix}\sim\begin{pmatrix}-13\\4\\2\end{pmatrix}$	**M1 A1**	Finds normal to the plane.
	$\dfrac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-13\\4\\2\end{pmatrix}}{\sqrt{1^2+1^2+0^2}\sqrt{13^2+4^2+2^2}}=-\dfrac{9}{\sqrt{378}}=-\dfrac{\sqrt{42}}{14}$	**M1 A1**	**FT** Uses dot product of their normal with direction of line.
	$\cos^{-1}\left(-\dfrac{\sqrt{42}}{14}\right)-90=27.6°\quad(\text{or } 0.481\,\text{rad})$	**A1**	
		5	

Question	Answer	Marks	Guidance
8	$9u^2+6u+1=0 \Rightarrow (3u+1)^2=0$	M1	Auxiliary equation.
	CF: $x=(At+B)e^{-\frac{1}{3}t}$	A1	States CF.
	$x=p\sin t+q\cos t \Rightarrow \dot{x}=p\cos t-q\sin t$ $\Rightarrow \ddot{x}=-p\sin t-q\cos t$	M1	Forms PI and differentiates.
	$-9p-6q+p=50$ $-9q+6p+q=0$	M1	Substitutes.
	$q=-3, p=-4.$	A1	
	$x=(At+B)e^{-\frac{1}{3}t}-4\sin t-3\cos t$	A1	States general solution. FT if correct form of CF/PI.
	$\dot{x}=-\frac{1}{3}(At+B)e^{-\frac{1}{3}t}+Ae^{-\frac{1}{3}t}-4\cos t+3\sin t$	M1	Differentiates and forms equations using initial conditions.
	$B=3$	B1	
	$-\frac{1}{3}B+A-4=0 \Rightarrow A=5$	A1	.
	$x=(5t+3)e^{-\frac{1}{3}t}-4\sin t-3\cos t$	A1	States PS.
		10	

Question	Answer	Marks	Guidance
9(i)	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma} \Rightarrow c=\alpha\beta+\beta\gamma+\alpha\gamma=5$	**M1 A1**	Uses $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$.
	$d=12$	**B1**	
		3	
9(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	**M1**	
	$(\alpha+\beta+\gamma)^2-10=b^2-10$	**A1**	
	Alternative method for 9(ii)		
	$90+bS_2-5b+36=0 \Rightarrow S_2=\frac{5b-126}{b}$	**M1 A1**	Sums over roots
		2	

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9(iii)	$x^3+bx^2+5x+12=0$ $(-b=\alpha+\beta+\gamma)$	M1	Formulates equation.
	$90+b\left(b^2-10\right)-5b+36=0$	M1 A1	Sums over roots and uses **9(ii)**
	$b^3-15b+126=0$	A1	
	Alternative method 1 for 9(iii)		
	$(\alpha+\beta+\gamma)^3=\alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma$	M1 A1	Uses expansion of $(\alpha+\beta+\gamma)^3$
	$-b^3=90-15b+36$	(M1)	Substitutes known values.
	$b^3-15b+126=0$	A1	
	Alternative method 2 for 9(iii)		
	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=b^2-10$	M1 A1	Finds sum of squares in terms of b.
	$\dfrac{5b-126}{b}=b^2-10\Rightarrow b^3-15b+126=0$	M1 A1	Equates expressions.
		4	
9(iv)	Real root is $b=-6$	M1 A1	Gives the real root for b.

Question	Answer	Marks	Guidance
10(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{n+1} x\right) = -(n+1)\cot^{n} x \csc^2 x$	**M1 A1**	Differentiates using chain rule.
	$= -(n+1)\cot^{n} x\left(\cot^2 x + 1\right)$ $= -(n+1)\cot^{n+2} x - (n+1)\cot^{n} x$	**M1**	Uses $\csc^2 x = \cot^2 x + 1$. OE
	$\left[\cot^{n+1} x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = -(n+1)\left(I_{n+2} + I_n\right)$	**M1**	Integrates both sides.
	$-1 = -(n+1)\left(I_{n+2} + I_n\right) \Rightarrow I_{n+2} = \frac{1}{n+1} - I_n$	**A1**	AG
10(i)	**Alternative method for 10(i)**		
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos^{n+1} x}{\sin^{n+1} x}\right) = \frac{-(n+1)\sin^{n+2} x\cos^{n} x - (n+1)\sin^{n} x\cos^{n+2} x}{\sin^{2n+2} x}$	**M1 A1**	Differentiates using quotient/chain rule.
	$-(n+1)\cot^{n} x - (n+1)\cot^{n+2} x$	**M1**	Separates
	$\left[\cot^{n+1} x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = -(n+1)\left(I_{n+2} + I_n\right)$	**M1**	Integrates both sides.
	$-1 = -(n+1)\left(I_{n+2} + I_n\right) \Rightarrow I_{n+2} = \frac{1}{n+1} - I_n$	**A1**	AG
		5	

Question	Answer	Marks	Guidance
10(ii)	$\bar{y} = \dfrac{1}{2A}\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^2 x \,\mathrm{d}x = \dfrac{I_2}{2A}$	**M1**	Uses correct formula for $\bar{y}$. Allow incorrect or missing limits.
	$A = \displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot x \,\mathrm{d}x = [\ln \sin x]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = \ln\sqrt{2}$	**M1 A1**	Integrates $\cot x$ to find area under the curve with correct limits.
	$I_2 = 1 - I_0 = 1 - \dfrac{1}{4}\pi$	**M1 A1**	Finds I_2
	$\bar{y} = \dfrac{1}{\ln 2}\left(1 - \dfrac{1}{4}\pi\right)$	**A1**	AEF. Must be exact.
		6	

Question	Answer	Marks	Guidance
11E(i)	$\mathbf{P}=\begin{pmatrix}1&-1&0\\1&0&1\\0&b&-1\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}2&0&0\\0&1&0\\0&0&3\end{pmatrix}$	**B1 B1**	Writes **P** and **D** (accept correctly matched permutations of columns)
	$\det\mathbf{P}=-b-1$	**B1**	
	$\mathbf{P}^{-1}=\frac{1}{b+1}\begin{pmatrix}b&-1&-b\\1&1&b\\1&1&-1\end{pmatrix}^{\mathrm{T}}=\frac{1}{b+1}\begin{pmatrix}b&1&1\\-1&1&1\\-b&b&-1\end{pmatrix}$	**M1 A1**	Finds inverse of **P**. (Adj $\div$ Det).
	$\mathbf{A}=\mathbf{PDP}^{-1}$	**M1**	Applies $\mathbf{A}=\mathbf{PDP}^{-1}$
	$\mathbf{A}=\frac{1}{b+1}\begin{pmatrix}1&-1&0\\1&0&1\\0&b&-1\end{pmatrix}\begin{pmatrix}2b&2&2\\-1&1&1\\-3b&3b&-3\end{pmatrix}$	**M1 A1**	Multiplies two adjacent matrices. $\mathbf{PD}=\begin{pmatrix}2&-1&0\\2&0&3\\0&b&-3\end{pmatrix}$
	$\frac{1}{b+1}\begin{pmatrix}1+2b&1&1\\-b&3b+2&-1\\2b&-2b&3+b\end{pmatrix}$	**A1**	

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11E(i)	**Alternative method for 11E(i)**		
	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$	**B3**	Matches eigenvalues with eigenvectors to form systems of equations. B1 for each correctly matched pair.
	Three values found correctly from solving systems of equations. Another three values found correctly from solving systems of equations. Attempting to find final three values from systems of equations.	**M1 A1** **M1 A1** **M1**	
	$\mathbf{A} = \frac{1}{b+1}\begin{pmatrix} 1+2b & 1 & 1 \\ -b & 3b+2 & -1 \\ 2b & -2b & 3+b \end{pmatrix}$	**A1**	Final answer given in matrix form.
		9	
11E(ii)	$\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	**M1 A1**	M1 for $\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
		2	

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11E(iii)	$\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=2^2\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow n=2$	**B1**	Uses $\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=2^n\begin{pmatrix}1\\1\\0\end{pmatrix}$
	$\mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b^{-1}\end{pmatrix}\Rightarrow b=b^{-1}\Rightarrow b=1$	**M1 A1**	M1 for stating $b=b^{-1}$ (since $\mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b\end{pmatrix}$). Must be only one answer ($b=1$) for A1.
		3	

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11O(i)(a)	$\ln y = t \ln a$ $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = \ln a$	**M1 A1**	Differentiates both sides.
	$\frac{\mathrm{d}y}{\mathrm{d}t} = y \ln a = a^t \ln a$	**A1**	AG
		3	
11O(i)(b)	$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = a^t (\ln a)^2$	**B1**	
11O(ii)	$r = \ln a$	**B1**	States common ratio.
	$-1 < \ln a < 1 \Rightarrow \mathrm{e}^{-1} < a < \mathrm{e}$	**M1 A1**	Uses convergence condition for a geometric series.
		3	

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11O(iii)	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{a^t\ln a}{at^{a-1}}=\left(\frac{\ln a}{a}\right)\left(\frac{a^t}{t^{a-1}}\right)$	**M1 A1**	Uses chain rule to find $\frac{dy}{dx}$
	$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\left(\frac{\ln a}{a}\right)\left(\frac{t^{a-1}a^t\ln a-(a-1)t^{a-2}a^t}{t^{2(a-1)}}\right)$	**M1 A1**	Uses quotient or product rule to find $\frac{d}{dt}\left(\frac{dy}{dx}\right)$
	$\frac{d^2y}{dx^2}=\left(\frac{\ln a}{a^2}\right)\left(\frac{t^{a-1}a^t\ln a-(a-1)t^{a-2}a^t}{t^{3(a-1)}}\right)$ $=\left(\frac{\ln a}{a^2}\right)\left(\frac{\ln a-(a-1)t^{-1}}{t^{2(a-1)}}\right)a^t$	**M1 A1**	Uses chain rule to find $\frac{d^2y}{dx^2}$
	$t=2\Rightarrow\frac{d^2y}{dx^2}=2^{1-2a}\ln a(2\ln a-a+1)$	**A1**	Substitutes $t=2$, AG.
		7	

